Domácí cvičení 11

(Riemannův integrál)

11/ 1) Vypočtěte:

a) $\int_a^b k\,\mathrm{d}x$ (k – konstanta), b) $\int_{-\sqrt{2}}^{\sqrt{2}} (x^3 - 3x^2 + 6x - 8)\,\mathrm{d}x$, c) $\int_0^{\frac{1}{2}} \frac{2}{\sqrt{1-x^2}}\,\mathrm{d}x$,

d) $\int_{-\frac{\pi}{2}}^{\pi} \cos 5x\,\mathrm{d}x$, e) $\int_{\pi}^{3\pi} \frac{1}{\sin^2 \frac{x}{4}}\,\mathrm{d}x$, f) $\int_3^5 \frac{1}{(x-2)^3}\,\mathrm{d}x$.

11/ 2) Vypočtěte:

a) $\int_{-1}^{1} |2x+1|\,\mathrm{d}x$, b) $\int_0^3 |x^2 - 3x + 2|\,\mathrm{d}x$, c) $\int_0^{2\pi} |\sin x|\,\mathrm{d}x$,

d) $\int_{-\pi}^{\pi} |\cos x|\,\mathrm{d}x$, e) $\int_{-2}^{1} \mathrm{e}^{|x|-3}\,\mathrm{d}x$.

11/ 3) Vypočtěte:

a) $\int_0^{\frac{\pi}{6}} (x+2)\sin 3x\,\mathrm{d}x$, b) $\int_1^{\mathrm{e}} \ln x\,\mathrm{d}x$, c) $\int_{-2}^{2} (x^2+1)\,\mathrm{e}^{\frac{x}{2}}\,\mathrm{d}x$.

11/ 4) Vypočtěte:

a) $\int_0^{\frac{1}{2}} \frac{x}{2x^2+3x+1}\,\mathrm{d}x$, b) $\int_{-3}^{-2} \frac{2}{x^4 - x^2}\,\mathrm{d}x$, c) $\int_1^2 \frac{4}{x^3+x}\,\mathrm{d}x$, d) $\int_{-2}^{0} \frac{3x^3+14x-2}{(x-1)(x^2+4)}\,\mathrm{d}x$.

11/ 5) Vypočtěte:

a) $\int_0^{\frac{\pi}{6}} \frac{\mathrm{d}x}{\cos x}$, b) $\int_{\ln 2}^{\ln 5} \frac{\mathrm{d}x}{\mathrm{e}^x - 1}$, c) $\int_{\mathrm{e}^{-1}}^{\mathrm{e}} \frac{\ln x + 1}{x(\ln^2 x + 1)}\,\mathrm{d}x$.

11/ 6) Vypočtěte:

a) $\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \sin^4 2x\,\mathrm{d}x$, b) $\int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \frac{4\arcsin x}{\sqrt{1-x^2}}\,\mathrm{d}x$, c) $\int_0^1 \frac{3x^2+4x+2}{\sqrt{x^3+2x^2+2x+4}}\,\mathrm{d}x$.

Výsledky:

V každém příkladu je potřeba ověřit existenci hledaného Riemannova integrálu. K tomu stačí, je-li integrovaná funkce spojitá na (uzavřeném) intervalu, přes který integrujeme. Zde toto platí v každém příkladu. Nebudu to tedy již uvádět u každého příkladu zvlášť (i když v písemce byste to uvedené mít měli).

I je opět hledaný integrál. Jako dříve také neuvádím úplný popis substituce, ale jen jakou soubstituci jsem použila.

11/ 1) a) $I = k(b-a)$ (nakreslete si obrázek a integrály z konstanty příště nepočítejte přes primitivní funkci!),

b) $I = \left[\frac{x^4}{4} - x^3 + 3x^2 - 8x\right]_{-\sqrt{2}}^{\sqrt{2}} = -20\sqrt{2}$, c) $I = [2\arcsin x]_0^{\frac{1}{2}} = \frac{\pi}{3}$, d) $I = \left[\frac{\sin 5x}{5}\right]_{-\frac{\pi}{2}}^{\pi} = \frac{1}{5}$,

e) $I = \left[-4\mathrm{cotg}\,\frac{x}{4}\right]_{\pi}^{3\pi} = 8$, f) $I = \left[-\frac{1}{2}\,\frac{1}{(x-2)^2}\right]_3^5 = \frac{4}{9}$.

11/ 2) a) $I = \int_{-1}^{-\frac{1}{2}} (-2x-1)\,dx + \int_{-\frac{1}{2}}^{1} (2x+1) = \frac{1}{4} + \frac{9}{4} = \frac{5}{2}$,

b) $I = \int_0^1 (x^2-3x+2)\,dx + \int_1^2 (-x^2+3x-2)\,dx + \int_2^3 (x^2-3x+2)\,dx = \frac{5}{6} + \frac{1}{6} + \frac{5}{6} = \frac{11}{6}$,

c) $I = \int_0^{\pi} \sin x\,dx + \int_{\pi}^{2\pi} (-\sin x)\,dx = 2+2 = 4$,

d) $I = \int_{-\pi}^{-\frac{\pi}{2}} (-\cos x)\,dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx + \int_{\frac{\pi}{2}}^{\pi} (-\cos x)\,dx = 1+2+1 = 4$,

e) $I = \int_{-2}^{0} e^{-x-3}\,dx + \int_0^1 e^{x-3}\,dx = (-e^{-3} + e^{-1}) + (e^{-2} - e^{-3}) = e^{-3}(e^2 + e - 2)$.

11/ 3) a) $I = \left[(x+2)(-\frac{1}{3}\cos 3x)\right]_0^{\frac{\pi}{6}} + \int_0^{\frac{\pi}{6}} \frac{1}{3}\cos 3x\,dx = \frac{2}{3} + \frac{1}{9} = \frac{7}{9}$,

b) $I = [x\ln x]_1^e - \int_1^e 1\,dx = e - (e-1) = 1$,

c) $I = \left[(x^2+1)2e^{\frac{x}{2}}\right]_{-2}^2 - 4\int_{-2}^2 x e^{\frac{x}{2}}\,dx = \left[(x^2+1)2e^{\frac{x}{2}}\right]_{-2}^2 - 4\left(\left[(x\cdot 2e^{\frac{x}{2}}\right]_{-2}^2 - 2\int_{-2}^2 e^{\frac{x}{2}}\,dx\right) =$

$= 10(e - e^{-1}) - 4\Big(4(e + e^{-1}) - 4(e - e^{-1})\Big) = 10e - 42e^{-1}$.

11/ 4) a) $I = \frac{1}{2}\int_0^{\frac{1}{2}} \frac{x}{(x+1)(x+\frac{1}{2})}\,dx = \frac{1}{2}\int_0^{\frac{1}{2}} \left(\frac{2}{x+1} - \frac{1}{x+\frac{1}{2}}\right)dx = \frac{1}{2}\left[2\ln|x+1| - \ln\left|x+\frac{1}{2}\right|\right]_0^{\frac{1}{2}} =$

$= \frac{1}{2}\left(\left(2\ln\frac{3}{2} - \ln 1\right) - \left(2\ln 1 - \ln\frac{1}{2}\right)\right) = \ln\frac{3}{2} + \frac{1}{2}\ln\frac{1}{2}$,

b) $I = \int_{-3}^{-2} \left(-\frac{2}{x^2} + \frac{1}{x-1} - \frac{1}{x+1}\right)dx = \left[\frac{2}{x} + \ln\left|\frac{x-1}{x+1}\right|\right]_{-3}^{-2} = (-1+\ln 3) - \left(-\frac{2}{3} + \ln 2\right) = -\frac{1}{3} + \ln\frac{3}{2}$,

c) $I = \int_1^2 \left(\frac{4}{x} - \frac{4x}{x^2+1}\right)dx = \left[4\ln|x| - 2\ln(x^2+1)\right]_1^2 = (4\ln 2 - 2\ln 5) - (4\ln 1 - 2\ln 2) = 6\ln 2 - 2\ln 5$,

d) $I = \int_{-2}^{0} \left(3 + \frac{3}{x-1} + \frac{2}{x^2+4}\right)dx = \left[3x + 3\ln|x-1| + \operatorname{arctg}\frac{x}{2}\right]_{-2}^0 = 0 - (-6 + 3\ln 3 + \operatorname{arctg}(-1)) =$

$= 6 - 3\ln 3 + \frac{\pi}{4}$.

11/ 5) a) $I = |t = \sin x| = \int_0^{\frac{1}{2}} \left(\frac{-\frac{1}{2}}{t-1} + \frac{\frac{1}{2}}{t+1}\right)dt = \left[-\frac{1}{2}\ln|t-1| + \frac{1}{2}\ln|t+1|\right]_0^{\frac{1}{2}} =$

$= \left(-\frac{1}{2}\ln\frac{1}{2} + \frac{1}{2}\ln\frac{3}{2}\right) - \left(-\frac{1}{2}\ln 1 + \frac{1}{2}\ln 1\right) = \frac{1}{2}\ln 3$,

b) $I = |t = e^x| = \int_2^5 \left(\frac{1}{t-1} - \frac{1}{t}\right)dt = [\ln|t-1| - \ln|t|]_2^5 = (\ln 4 - \ln 5) - (\ln 1 - \ln 2) = 3\ln 2 - \ln 5$,

c) $I = |t = \ln x| = \int_{-1}^1 \left(\frac{1}{2}\frac{2t}{t^2+1} + \frac{1}{t^2+1}\right)dt = \left[\frac{1}{2}\ln|t^2+1| + \operatorname{arctg} t\right]_{-1}^1 =$

$= \left(\frac{1}{2}\ln 2 + \operatorname{arctg} 1\right) - \left(\frac{1}{2}\ln 2 + \operatorname{arctg}(-1)\right) = \frac{\pi}{2}$.

11/ 6) a) $I = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1}{4}(1-\cos 4x)^2\,dx = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \left(\frac{1}{4} - \frac{1}{2}\cos 4x + \frac{1}{8}(1+\cos 8x)\right)dx = \left[\frac{3}{8}x - \frac{1}{8}\sin 4x + \frac{1}{64}\sin 8x\right]_{\frac{\pi}{8}}^{\frac{3\pi}{8}} =$

$= \frac{3\pi}{32} + \frac{1}{4}$,

b) $I = |t = \arcsin x| = \int_{-\frac{\pi}{4}}^{\frac{\pi}{3}} 4t\,dt = [2t^2]_{-\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{7\pi^2}{72}$,

c) $I = |t = x^3 + 2x^2 + 2x + 4| = \int_4^9 \frac{1}{\sqrt{t}}\,dt = [2\sqrt{t}]_4^9 = 2$.