Určitý integrál - příklady

UKÁZKA

$$\int_1^2 x^2\,dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

$$\int_0^4 \sqrt{x}\,dx = \left[\frac{x^{3/2}}{3/2}\right]_0^4 = \frac{16}{3}$$

3 $$\int_0^{\pi} \sin x\,dx = \left[-\cos x\right]_0^{\pi} = 2$$

4 $$\int_{-2}^{1} \frac{dt}{t^2+1} = \left[\operatorname{arctg} t\right]_{-2}^{1} = \frac{\pi}{4} + \operatorname{arctg} 2$$

lichá fce

x $$\int_0^1 \left(\frac{1}{\sqrt{x^2+3}} - \frac{2}{x+1} + \frac{4}{x^2+2} + \frac{x}{x^2+2}\right)dx = \ln 3 - \frac{5}{2}\ln 2 + 2\sqrt{2}\operatorname{arctg}\frac{1}{\sqrt{2}}$$

PER PARTES

$$\int_a^b u(x)\,v'(x)\,dx = \left[u(x)v(x)\right]_a^b - \int_a^b u'(x)\,v(x)\,dx$$

8 $$\int_1^2 (x^2+1)\ln x\,dx = \left| \begin{matrix} u = \ln x & v' = x^2+1 \\ u' = \frac{1}{x} & v = \frac{1}{3}x^3 + x \end{matrix} \right| \ldots = \frac{14}{3}\ln 2 - \frac{16}{9}$$

9 $$\int_0^{\pi} x^2\cos x\,dx = \left| \begin{matrix} u = x^2 & v' = \cos x \\ u' = 2x & v = \sin x \end{matrix} \right| = \ldots = -2\pi$$

SUBSTITUCE

musíme změnit i meze! nebo dosadit zpět ze substituce

$$\int_0^1 x(x^2-1)^3\,dx = \left| \begin{matrix} x^2 - 1 = t \\ 2x\,dx = dt \\ dx = \frac{dt}{2x} \\ 1 \underset{1^2-1}{\rightsquigarrow} 0,\ 0 \underset{0^2-1}{\rightsquigarrow} -1 \end{matrix} \right| = \int_{-1}^{0} \frac{1}{2}t^3\,dt = \frac{1}{2}\left[\frac{t^4}{4}\right]_1^0 = -\frac{1}{8}$$

$$\int_{\pi}^{2\pi} e^{\sin x}\cdot\cos x\,dx = \left| \begin{matrix} \sin x = t \\ \cos x\,dx = dt \\ dx = \frac{dt}{\cos x} \\ \pi \underset{\sin\pi}{\rightsquigarrow} 0,\ 2\pi \underset{\sin 2\pi}{\rightsquigarrow} 0 \end{matrix} \right| = \int_0^0 e^t\,dt = 0$$

$$\int_0^{\pi/2} \frac{\sin x \cos^2 x}{\sqrt[4]{1+\cos^3 x}}\,dx = \left| \begin{array}{l} 1+\cos^3 x = u \\ -3\cos^2 x \cdot \sin x\, dx = du \\ dx = -du / 3\cos^2 x \cdot \sin x \\ 0 \rightsquigarrow 2,\ \frac{\pi}{2} \rightsquigarrow 1 \end{array}\right| = \int_2^1 \frac{-1/3}{\sqrt[4]{u}}\,du =$$

$$= \frac{4}{9}\left(\sqrt[4]{8}-1\right)$$

$$\int_1^4 \frac{\sqrt{x}-1}{\sqrt{x}+1}\,dx = \left| \begin{array}{l} u = \sqrt{x} \\ du = \frac{1}{2}dx \\ dx = 2du \\ 1 \rightsquigarrow 1,\ 4 \rightsquigarrow 2 \end{array}\right| = \int_1^2 \frac{u-1}{u+1}\,2u\,du = \int_1^2 \frac{2u^2-2u}{u+1}\,du =$$

vydělíme polynom

$$= \int_1^2 \left(\frac{4}{u+1}+2u-4\right)du = \left[u^2-4u+4\ln|u+1|\right]_1^2 = 4\ln 3 - 4\ln 2 - 1$$

$$\int_0^1 \frac{dx}{\sqrt{x^2+1}-x} = \left| \begin{array}{l} t = \sqrt{x^2+1} - x \Rightarrow x = \frac{1-t^2}{2t} \qquad 1 \rightsquigarrow \sqrt{2}-1 \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad 0 \rightsquigarrow 1 \\ \cancel{dt = \left(\frac{1}{2}\cdot 2x \cdot \frac{1}{\sqrt{x^2+1}} - 1\right)dx} \\ dx = -\frac{t^2+1}{2t^2}\,dt \end{array}\right| \qquad \begin{array}{l} t + x = \sqrt{x^2+1} \\ t^2 + 2tx + \cancel{x^2} = \cancel{x^2} + 1 \end{array}$$

$$= \int_1^{\sqrt{2}-1} \frac{1}{t}\cdot\left(-\frac{t^2+1}{2t^2}\right)dt = \int_{\sqrt{2}-1}^1 \frac{t^2+1}{2t^3}\,dt = -\frac{1}{4} - \frac{1}{2}\ln(\sqrt{2}-1) + \frac{1}{4(3-2\sqrt{2})}$$

můžeme změnit znaménko

$$= \frac{1+\sqrt{2}}{2} - \frac{1}{2}\ln(\sqrt{2}-1)$$

$$\int_0^1 \sqrt{x^2+1}\,dx = \left| \begin{array}{l} x = \cot t \\ dx = -\frac{1}{\sin^2 t}\,dt \\ 0 \rightsquigarrow \frac{\pi}{2},\ 1 \rightsquigarrow \frac{\pi}{4} \end{array}\right| = \int_{\pi/2}^{\pi/4} \sqrt{\frac{\cos^2 t}{\sin^2 t}+1}\cdot\frac{-1}{\sin^2 t}\,dt =$$

$$= \int_{\pi/4}^{\pi/2} \frac{1}{|\sin t|}\cdot\frac{1}{\sin^2 t}\,dt = \int_{\pi/4}^{\pi/2} \frac{1}{\sin^3 t}\,dt$$

Určitý integrál

PER PARTES

$$\int_0^{\frac{\pi}{6}} (x+2)\sin 3x\,dx = \begin{vmatrix} u = x+2 & v' = \sin 3x \\ u' = 1 & v = -\frac{1}{3}\cos 3x \end{vmatrix} = \left[(x+2)\left(-\frac{1}{3}\cos 3x\right)\right]_0^{\pi/6} + \int_0^{\pi/6} \frac{1}{3}\cos 3x\,dx =$$

$$= \frac{2}{3} + \frac{1}{9} = \underline{\underline{\frac{7}{9}}}$$

$$\int_1^e \ln x\,dx = \begin{vmatrix} u = \ln x & v' = 1 \\ u' = \frac{1}{x} & v = x \end{vmatrix} = \left[x \ln x\right]_1^e - \int_1^e 1\,dx = \underline{\underline{1}}$$

$$\int_{-2}^{2} (x^2+1)e^{\frac{x}{2}}\,dx = \begin{vmatrix} u = x^2+1 & v' = e^{\frac{x}{2}} \\ u' = 2x & v = 2e^{\frac{x}{2}} \end{vmatrix} = \left[(x^2+1)2e^{\frac{x}{2}}\right]_{-2}^{2} - 4\int_{-2}^{2} x e^{\frac{x}{2}}\,dx =$$

$$= \begin{vmatrix} u = x & v' = e^{\frac{x}{2}} \\ u' = 1 & v = 2e^{\frac{x}{2}} \end{vmatrix} = \left[(x^2+1)2e^{\frac{x}{2}}\right]_{-2}^{2} - 4\left(\left[x\, 2e^{x/2}\right]_{-2}^{2} - 2\int_{-2}^{2} e^{\frac{x}{2}}\,dx\right) =$$

$$= 10e - 42e^{-1}$$

$$\int_1^2 (x^2+1)\ln x\,dx = \begin{vmatrix} u = \ln x & v' = x^2+1 \\ u' = \frac{1}{x} & v = \frac{1}{3}x^3 + x \end{vmatrix} = \dots = \frac{14}{3}\ln 2 - \frac{16}{9}$$

$$\int_0^{\pi} x^2 \cos x\,dx = \begin{vmatrix} u = x^2 & v' = \cos x \\ u' = 2x & v = \sin x \end{vmatrix} = \dots = -2\pi$$

SUBSTITUCE

$$\int_0^{\pi/6} \frac{dx}{\cos x} = \left| \begin{array}{l} t = \sin x \\ dt = \cos x\, dx \\ \frac{\pi}{6} \rightsquigarrow \frac{1}{2} \\ 0 \rightsquigarrow 0 \end{array} \right| = \dots = \int_0^{1/2} \left(\frac{-\frac{1}{2}}{t-1} + \frac{\frac{1}{2}}{t+1} \right) dt = \left[-\frac{1}{2}\ln|t-1| + \frac{1}{2}\ln|t+1| \right]_0^{1/2} =$$

$$= \frac{1}{2}\ln 3$$

$$\int_{\ln 2}^{\ln 5} \frac{dx}{e^x - 1} = \left| \begin{array}{l} t = e^x \\ dt = e^x\, dx \\ \ln 5 \rightsquigarrow 5 \\ \cancel{\ln 5}\, e^{\ln 5} = t \\ t = 5 \\ \ln 2 \rightsquigarrow 2 \end{array} \right| = \int_2^5 \left(\frac{1}{t-1} - \frac{1}{t} \right) dt = 3\ln 2 - \ln 5$$

$$\int_{e^{-1}}^{e} \frac{\ln x + 1}{x(\ln^2 x + 1)}\, dx = \left| \begin{array}{l} t = \ln x \\ dt = \frac{1}{x} dx \\ t = \ln e = 1 \\ e \rightsquigarrow 1 \\ e^{-1} \rightsquigarrow -1 \end{array} \right| = \int_{-1}^{1} \left(\frac{1}{2} \frac{2t}{t^2+1} + \frac{1}{t^2+1} \right) dt = \left[\frac{1}{2}\ln|t^2+1| + \operatorname{arctg} t \right]_{-1}^{1}$$

$$= \frac{\pi}{2}$$