Určitý integrál : substituce

Př 1

$$\int_3^8 \frac{\sqrt{x+1}+1}{\sqrt{x+1}-1}\,dx = \left| \begin{array}{l} \sqrt{x+1} = t \\ \frac{1}{2\sqrt{x+1}} = \frac{dt}{dx} \\ dx = 2\sqrt{x+1}\,dt \\ \quad = 2t\,dt \end{array} \right| = \int_2^3 \frac{t+1}{t-1}\,2t\,dt = 2\int_2^3 \frac{t^2+t}{t-1}\,dt =$$

$$(t^2+t):(t-1) = t+2+\frac{2}{t-1}$$
$$\frac{-(t^2-t)}{2t}$$
$$\frac{-(2t-2)}{2}$$

přepočet mezí: $x=3 \Rightarrow \underline{t} = \sqrt{3+1} = \underline{2}$

$x=8 \Rightarrow \underline{t} = \sqrt{8+1} = \underline{3}$

$$= 2\int_2^3 \left(t+2+\frac{2}{t-1}\right)dt = 2\left[\frac{t^2}{2}+2t+2\cdot\ln|t-1|\right]_2^3 = 2\cdot\left\{\left[\frac{3^2}{2}+2\cdot 3+2\cdot\ln|3-1|\right] - \right.$$

$$\left. - \left[\frac{2^2}{2}+2\cdot 2+2\cdot\ln|2-1|\right]\right\} = 2\cdot\left\{\left[\frac{9}{2}+\cancel{6}+2\ln 2-\cancel{6}\right]\right\} = \underline{\underline{9+4\ln 2}}$$

Př 2

$$\int_{-\infty}^{\infty} \frac{1}{x^2+x+1}\,dx = \int_{-\infty}^{0} \frac{1}{x^2+x+1}\,dx + \int_{0}^{\infty} \frac{1}{x^2+x+1}\,dx = \lim_{y\to-\infty} \int_y^0 \frac{1}{x^2+x+1}\,dx +$$

$$+ \lim_{y\to\infty} \int_0^y \frac{1}{x^2+x+1}\,dx = \lim_{y\to-\infty} \int_y^0 \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx + \lim_{y\to\infty} \int_0^y \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx =$$

$$\left| \begin{array}{l} x^2+x+1 = x^2+2\cdot\frac{1}{2}x+\frac{1}{4}-\frac{1}{4}+1 \\ \quad = (x+\frac{1}{2})^2+\frac{3}{4} \end{array} \right|$$

$$= \left| \begin{array}{l} (x+\frac{1}{2})^2 = \frac{3}{4}t^2 \\ \frac{4}{3}(x+\frac{1}{2})^2 = t^2 \\ \sqrt{\frac{4}{3}}(x+\frac{1}{2}) = t \\ \sqrt{\frac{4}{3}} = \frac{dt}{dx} \Rightarrow dx = \frac{\sqrt{3}}{2}\,dt \end{array} \right| = \lim_{y\to-\infty} \int_{\sqrt{\frac{4}{3}}(y+\frac{1}{2})}^{\sqrt{\frac{1}{3}}} \frac{1}{\frac{3}{4}t^2+\frac{3}{4}}\cdot\frac{\sqrt{3}}{2}\,dt + \lim_{y\to\infty} \int_{\sqrt{\frac{1}{3}}}^{\sqrt{\frac{4}{3}}(y+\frac{1}{2})} \frac{1}{\frac{3}{4}t^2+\frac{3}{4}}\frac{\sqrt{3}}{2}\,dt =$$

Meze: $x+\frac{1}{2} = \sqrt{\frac{3}{4}}\,t$

$x = \sqrt{\frac{3}{4}}\,t - \frac{1}{2}$

$x \leftrightarrow y \rightarrow y = \sqrt{\frac{3}{4}}\,t - \frac{1}{2}$

$t = \sqrt{\frac{4}{3}}(x+\frac{1}{2})$: $x=0 \Rightarrow \underline{t = \sqrt{\frac{1}{3}}} =: b$

$x \to \pm\infty \Rightarrow \underline{t = \sqrt{\frac{4}{3}}(y+\frac{1}{2})} =: a$

(tj. $y\to\pm\infty$)

$$= \lim_{y\to-\infty} \int_a^b \frac{\cancel{4}^2}{3}\cdot\frac{\sqrt{3}}{\cancel{2}}\cdot\frac{1}{t^2+1}\,dt + \lim_{y\to\infty} \int_a^b \frac{\cancel{4}^2}{3}\cdot\frac{\sqrt{3}}{\cancel{2}}\cdot\frac{1}{t^2+1}\,dt = \frac{2\sqrt{3}}{3}\left\{\lim_{y\to-\infty} \int_a^b \frac{1}{t^2+1}\,dt + \right.$$

$$\left. + \lim_{y\to\infty} \int_a^b \frac{1}{t^2+1}\,dt \right\} = \frac{2\sqrt{3}}{3}\left\{\lim_{y\to-\infty}\left[\operatorname{arctg} t\right]_a^b + \lim_{y\to\infty}\left[\operatorname{arctg}\right]_a^b\right\} =$$

$$\frac{2\sqrt{3}}{3}\left\{\operatorname{arctg}\sqrt{\frac{1}{3}} - \underbrace{\lim_{y\to-\infty}\left[\operatorname{arctg}\sqrt{\frac{4}{3}}(y+\frac{1}{2})\right]}_{(-\frac{\pi}{2})} + \operatorname{arctg}\sqrt{\frac{1}{3}} - \underbrace{\lim_{y\to\infty}\left[\operatorname{arctg}\sqrt{\frac{4}{3}}(y+\frac{1}{2})\right]}_{\frac{\pi}{2}}\right\} =$$

$$= \frac{2\sqrt{3}}{3}\left\{2\operatorname{arctg}\sqrt{\frac{1}{3}} \cancel{-(-\frac{\pi}{2})} \cancel{-\frac{\pi}{2}}\right\} = \frac{4}{3}\operatorname{arctg}\frac{1}{\sqrt{3}} = \frac{4}{3}\operatorname{arctg}\frac{\sqrt{3}}{3} = \frac{4}{3}\cdot\frac{\pi}{6} = \frac{2\pi}{9}$$

Integrace: užití parciálních zlomků:

Př 1

$$\int \frac{x}{(x+1)(2x+1)}\,dx = \int \frac{1}{x+1}\,dx - \int \frac{1}{2x+1}\,dx = \underline{\underline{\ln|x+1| + \tfrac{1}{2}\ln|2x+1| + C}}$$

$$\frac{x}{(x+1)(2x+1)} = \frac{A}{x+1} + \frac{B}{2x+1} = \frac{A(2x+1)+B(x+1)}{(x+1)(2x+1)} = \frac{x(2A+B)+(A+B)}{(x+1)(2x+1)}$$

$$1 = 2A+B \qquad 1 = -2B+B$$
$$0 = A+B \Rightarrow A = -B \qquad 1 = -B \Rightarrow B = -1$$
$$A = 1$$

Př 2

$$\int \frac{x^3+1}{x^3-x^2}\,dx = \int 1\,dx + \int \frac{x^2+1}{x^3-x^2}\,dx = x - \int \frac{1}{x}\,dx - \underbrace{\int \frac{1}{x^2}\,dx}_{\int x^{-2}dx = \frac{x^{-1}}{-1}} + \int \frac{2}{x-1}\,dx =$$

$$(x^3+1):(x^3-x^2) = 1 + \frac{x^2+1}{x^3-x^2}$$
$$\underline{-(x^3-x^2)}$$
$$x^2+1$$

$$\frac{x^2+1}{x^3-x^2} = \frac{x^2+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} = \frac{Ax(x-1)+B(x-1)+Cx^2}{x^2(x-1)} = \frac{x^2(A+C)+x(-A+B)+(-B)}{x^2(x-1)}$$

$$\Rightarrow \quad 1 = A+C \Rightarrow \underline{C} = 1-A = 1+1 = \underline{2}$$
$$0 = -A+B \Rightarrow A = B \Rightarrow \underline{A = -1}$$
$$1 = -B \Rightarrow \underline{B = -1}$$

$$= \underline{\underline{x - \ln|x| + \frac{1}{x} + 2\ln|x-1| + C}}$$

DÚ:

$$\int \frac{1}{1+x^3}\,dx$$

$$\int_0^1 x^2 e^x\,dx$$

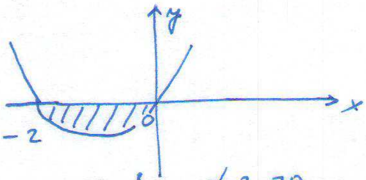
APLIKACE INT. POČTU

a) Obsah obrazce

Př: Vypočítej obsah obrazce, jenž je vymezen křivkami: $y = x^2 + 2x$ a osou x.

$y = x^2 + 2x = 0$

$x(x+2) = 0 \Rightarrow x = 0 \lor x = -2$

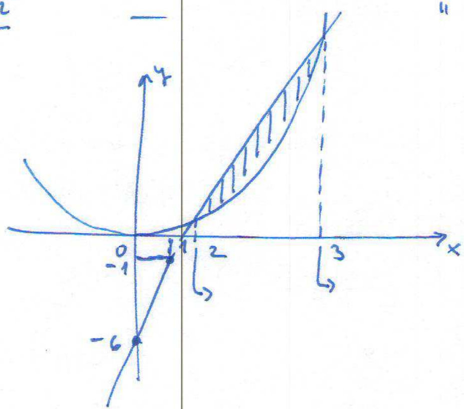


osa x: $y = 0$

$$\int_{-2}^{0} 0 - (x^2 + 2x)\,dx = -\int_{-2}^{0} (x^2 + 2x)\,dx = -\left[\frac{x^3}{3} + \frac{\cancel{2}x^2}{\cancel{2}}\right]_{-2}^{0} = +\left(\frac{-8}{3} + 4\right) = +\frac{4}{3}$$

$$= -\left[\frac{0}{3} + 0 - \left(\frac{(-2)^3}{3} + (-2)^2\right)\right]$$

Př: ″ — ″ — : $y = x^2$ a $y = 5x - 6$.



$x^2 = 5x - 6$

$x^2 - 5x + 6 = 0$

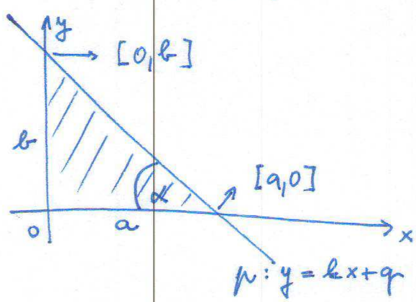
$2 \cdot 3$

$x_1 = 2;\ x_2 = 3$

$$\int_{2}^{3} (5x - 6 - x^2)\,dx = \left[\frac{5x^2}{2} - 6x - \frac{x^3}{3}\right]_2^3 = \frac{5 \cdot 9}{2} - 18 - 9 -$$

$$-\left(10 - 12 - \frac{8}{3}\right) = \frac{45}{2} + \frac{8}{3} - 25 = \frac{1}{6}$$

Př: Odvoďte vzorec pro obsah ◺.



$k = -\operatorname{tg}\alpha = -\frac{b}{a}$

$p: y = -\frac{b}{a}x + q$

$[a,0] \in p \Rightarrow 0 = -\frac{b}{\cancel{a}} \cdot \cancel{a} + q$

$b = q$

$p: y = -\frac{b}{a}x + b$

$$\int_0^a \left(-\frac{b}{a}x + b\right)dx = \left[-\frac{b}{a}\frac{x^2}{2} + bx\right]_0^a = -\frac{b}{\cancel{a}} \cdot \frac{a^{\cancel{2}}}{2} + ba = -\frac{ab}{2} + ab = \frac{ab}{2}.$$

DÚ: Vyp. S obrazce vymezeného křivkami:

a) $y = \sqrt{x}$ na $x \in \langle 0; 4\rangle$ $\left[\frac{16}{3}\right]$

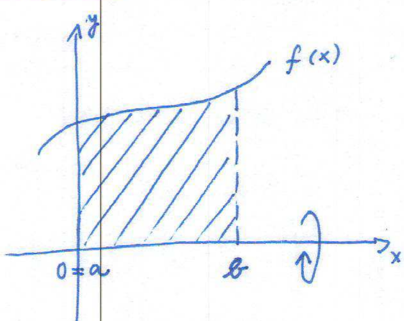
b) $y = \sin x$ $x \in \langle 0; \pi\rangle$ $[2]$

c) $y = -x^2 + 4$ a osa x $\left[\frac{32}{3}\right]$

d) $y = x^2 - 2x + 3$ a $y = -2x^2 + 4x + 3$ $[4]$

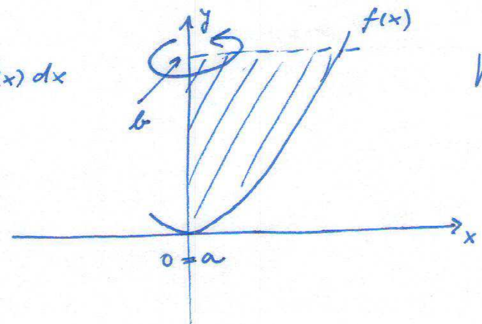
b) Objem rotačních těles

a)



rotace kolem osy x

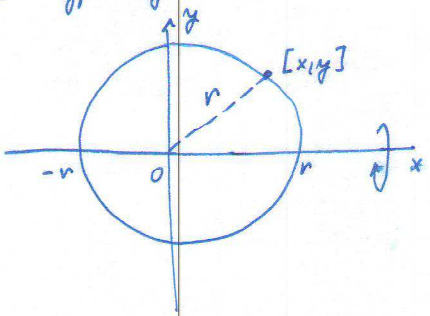
$$V = \pi \int_a^b f^2(x)\,dx$$



rotace kolem osy y

$$V = \pi \int_a^b f^2(y)\,dy$$

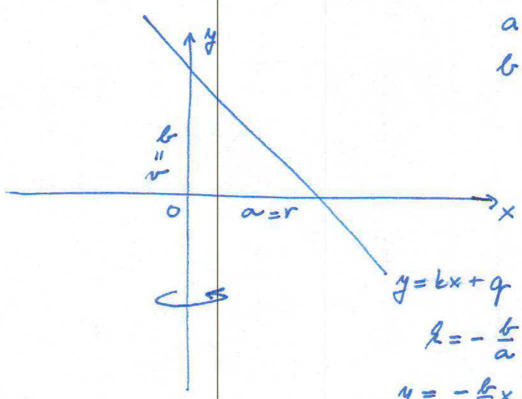
Př Vyp. objem koule.



$$r^2 = x^2 + y^2$$
$$y^2 = r^2 - x^2$$

$$V = \pi \int_{-r}^{r} (r^2 - x^2)\,dx = \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r} = \pi \left[r^3 - \frac{r^3}{3} - \left(-r^3 + \frac{r^3}{3}\right)\right] = \pi \left(2r^3 - \frac{2r^3}{3}\right) = \underline{\underline{\frac{4\pi r^3}{3}}}$$

Př Vyp. objem rotačního kužele.



a ... poloměr r

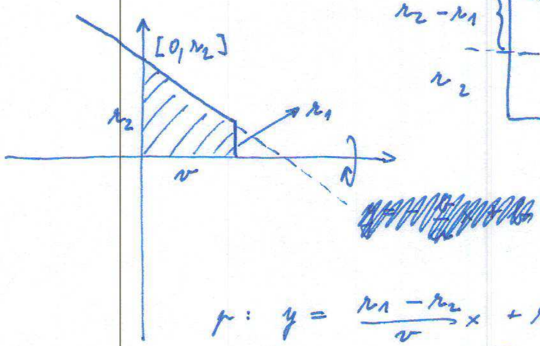
b ... výška v

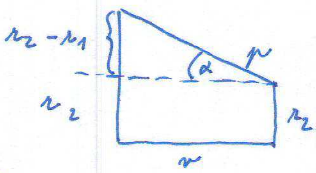
$$k = -\frac{b}{a}\ ;\ q = b$$

$$y = -\frac{b}{a}x + b \Rightarrow x = \frac{(y-b)\cdot a}{-b}$$

$$V = \pi \int_0^{v} \left[-\frac{a(y-b)}{b}\right]^2 dy = \pi \int_0^b \frac{a^2}{b^2}\cdot (y-b)^2\,dy =$$
$$= \pi \left(\frac{a}{b}\right)^2 \int_0^b (y^2 - 2yb + b^2)\,dy = \pi\left(\frac{a}{b}\right)^2 \left[\frac{y^3}{3} - \frac{2y^2 b}{2} + b^2 y\right]_0^b =$$
$$= \pi \left(\frac{a}{b}\right)^2 \left[\frac{b^3}{3} - \cancel{b^3} + \cancel{b^3}\right] =$$
$$= \pi \frac{a^2}{\cancel{b^2}} \cdot \frac{b^3}{3} = \underline{\underline{\frac{\pi a^2 b}{3}}}$$

Př Objem komolého kužele.





$$k = -\operatorname{tg}\alpha = -\frac{r_2 - r_1}{v}$$

$$[0, r_2] \in p:\quad y = -\frac{r_2 - r_1}{v}x + q$$
$$r_2 = \frac{-(r_2 - r_1)}{v}\cdot 0 + q$$
$$q = r_2$$

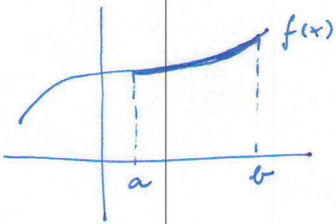
$$p:\ y = \frac{r_1 - r_2}{v}x + r_2$$

$$V = \pi \int_0^v \left(\frac{r_1 - r_2}{v}x + r_2\right)^2 dx = \pi \int_0^v \left(\frac{(r_1 - r_2)^2}{v^2}\cdot x^2 + \frac{2r_2(r_1 - r_2)}{v}x + r_2^2\right) dx =$$
$$= \pi \left[\left(\frac{r_1 - r_2}{v}\right)^2 \frac{x^3}{3} + \frac{2r_2(r_1 - r_2)}{v}\,\frac{x^2}{2} + r_2^2 x\right]_0^v = \pi\, \frac{(r_1 - r_2)^2}{\cancel{v^2}}\cdot\frac{v^3}{3} + \frac{\cancel{2}r_2(r_1 - r_2)}{\cancel{v}}\,\frac{v^{\cancel{2}}}{\cancel{2}} +$$
$$+ r_2^2 v = \pi v\left[\frac{(r_1 - r_2)^2}{3} + r_2(r_1 - r_2) + r_2^2\right] = \pi v\left[\frac{(r_1^2 - 2r_1 r_2 + r_2^2)}{3} + r_1 r_2 -\right.$$
$$\left. - \cancel{r_2^2} + \cancel{r_2^2}\right] = \frac{\pi v}{3}\left[r_1^2 - 2r_1 r_2 + r_2^2 + 3r_1 r_2\right] = \underline{\underline{\frac{\pi v}{3}(r_1^2 + r_1 r_2 + r_2^2)}}$$

DÚ: Vyp. objem: a) rotační elipsoid $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right)$

b) rotační paraboloid $(y = \sqrt{x})$

c) Délka křivky



$$L = \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

Př: Vyp. délku oblouku křivky $y = x^2-2x+3$ pro $0 \leq x \leq 1$.

$f(x) = x^2-2x+3 \Rightarrow f'(x) = 2x-2$

$$L = \int_0^1 \sqrt{1+(2x-2)^2}\,dx = \int_{-2}^0 \frac{1}{2}\sqrt{1+t^2}\,dt = \left| \begin{array}{l} t = \operatorname{tg} z \Rightarrow \operatorname{arctg} t = z \\ dt = \frac{1}{\cos^2 z}\,dz \\ z_D = \operatorname{arctg}(-2) \\ z_H = 0 \end{array} \right| =$$

$$\left| \begin{array}{l} 2x-2 = t \\ 2dx = dt \\ dx = \frac{1}{2}dt \end{array} \right|$$

$t_D = -2$

$t_H = 0$

$$= \frac{1}{2}\int_{\operatorname{arctg}(-2)}^0 \sqrt{1+\operatorname{tg}^2 z}\cdot\frac{1}{\cos^2 z}\,dz = \frac{1}{2}\int_{\operatorname{arctg}(-2)}^0 \sqrt{1+\frac{\sin^2 z}{\cos^2 z}}\,\frac{1}{\cos^2 z}\,dz =$$

$\operatorname{tg}^2 z = \frac{\sin^2 z}{\cos^2 z}$

$$= \frac{1}{2}\int_{\operatorname{arctg}(-2)}^0 \sqrt{\frac{\cos^2 z+\sin^2 z}{\cos^2 z}}\cdot\frac{1}{\cos^2 z}\,dz = \frac{1}{2}\int_{\operatorname{arctg}(-2)}^0 \frac{1}{\cos^3 z}\,dz = \frac{1}{2}\int_{\operatorname{arctg}(-2)}^0 \frac{\cos z}{\cos^4 z}\,dz =$$

$\cos^2 z+\sin^2 z = 1 \Rightarrow \sqrt{\frac{1}{\cos^2 z}} = \frac{1}{\cos z}$

$$= \frac{1}{2}\int_{\operatorname{arctg}(-2)}^0 \frac{\cos z}{(1-\sin^2 z)^2}\,dz = \left| \begin{array}{l} u = \sin z \\ du = \cos z\,dz \\ dz = \frac{1}{\cos z}\,du \end{array} \right| = \frac{1}{2}\int_{\sin(\operatorname{arctg}(-2))}^0 \frac{du}{(1-u^2)^2} =$$

$u_D = \sin(\operatorname{arctg}(-2))$

$u_H = 0$

$$\int \frac{du}{(1-u^2)^2} \overset{c}{=} \int \frac{A}{(1-u)^2} + \int \frac{B}{1-u} + \int \frac{C}{(1+u)^2} + \int \frac{D}{1+u}.$$

$(1-u^2)^2 = (1-u^2)(1-u^2)$

$= (1-u)^2(1+u)^2$

$$\frac{1}{(1-u^2)^2} = \frac{A\underbrace{(1+u)^2}_{1+2u+u^2} + B(1-u)(1+u)^2 + C(1-u)^2 + D(1+u)\underbrace{(1-u)^2}_{1-2u+u^2}}{(1-u)^2(1+u)^2} =$$

$$= \frac{A(1+2u+u^2) + B(1+2u+u^2-u-2u^2-u^3) + C(1-2u+u^2) + D(1-2u+u^2+u-2u^2+u^3)}{(1-u)^2(1+u)^2}$$

$\Rightarrow$

$1 = A+B+C+D$

$0 = 2A+B-2C-D$

$0 = A-B+C-D$

$0 = -B+D \Rightarrow B = D$

$\Rightarrow$

$1 = A+2B+C$

$0 = 2A-2C \Rightarrow A = C$

$0 = A-2B+C$

$1 = 2A+2B \Rightarrow 1-2B = 2B$

$0 = 2A-2B \Rightarrow A = B$

$1 = 4B \Rightarrow B = \frac{1}{4}$

$A = C$; $A = B$; $B = D$

$B = \frac{1}{4} \Rightarrow A = \frac{1}{4} \Rightarrow C = \frac{1}{4} \Rightarrow D = \frac{1}{4}.$

$$\int \frac{du}{(1-u^2)^2} \overset{c}{=} \frac{1}{4}\left[\int \frac{1}{(1-u)^2}du + \int \frac{1}{1-u}du + \int \frac{1}{(1+u)^2}du + \int \frac{1}{1+u}du\right] =$$

$$\overset{c}{=} \left| \begin{array}{l} 1-u = w \\ -du = dw \end{array} \right| \qquad \left| \begin{array}{l} 1+u = v \\ du = dv \end{array} \right|$$

$$\overset{c}{=} \frac{1}{4}\left[-\int \frac{1}{w^2}dw + \int \frac{1}{1-u}du + \int \frac{1}{v^2}dv + \int \frac{1}{1+u}du\right] =$$

$$\overset{c}{=} \frac{1}{4}\left[+\frac{1}{w} - \ln(1-u) - \frac{1}{v} + \ln(1+u)\right]$$

$$\overset{c}{=} \frac{1}{4}\left[\frac{1}{1-u} \underbrace{- \ln(1-u)}_{\ln(1-u)^{-1}} - \frac{1}{1+u} + \ln(u+1)\right]$$

$$\Rightarrow \frac{1}{2}\int_{\sin(\operatorname{arctg}(-2))}^{0} \frac{du}{(1-u^2)^2} = \frac{1}{2}\cdot\frac{1}{4}\left[\frac{1}{1-u} + \underbrace{\cancel{\ln(1-u)}}_{\ln(1-u)^{-1}} - \frac{1}{1+u} + \ln(u+1)\right]_{\underbrace{\sin(\operatorname{arctg}(-2))}_{\overset{!!}{k}}}^{0}$$

$$= \frac{1}{8}\left(-\frac{1}{1-k} - \ln(1-k)^{-1} + \frac{1}{1+k} - \ln(k+1)\right)$$