CVIČENÍ 11

Dif. rovnice se separovanými proměnnými

1) $2y - x^3 y' = 0$

$y \neq 0,\ x \neq 0$ ⟵ podmínky, pak třeba $y=0$ zkontrolovat na singulární řeš.

$x^3 y' = 2y$

$x^3 \frac{dy}{dx} = 2y$

$\frac{1}{y}\,dy = \frac{2}{x^3}\,dx$

$\ln|y| = 2 \cdot \frac{1}{x^2}\left(-\frac{1}{2}\right) + c, \quad c \in R$

$|y| = e^{-\frac{1}{x^2}} \cdot K, \quad K = e^c \Rightarrow K > 0$

$y = e^{-\frac{1}{x^2}} K_1, \quad K_1 \in R \setminus \{0\}$

$y=0$ je sice řeš., ale lze ho dosáhnout volbou $K_1 = 0$, a proto to není singulární řeš.

Singulární řeš.: řeš., které nelze vypočítat žádnou ~~ne~~ volbou integrační konstanty. (Většinou se získá z podmínek, které jsme v průběhu řešení postupně přidávali).

2) $(x+1)\,dy + xy\,dx = 0$

$$(x+1)\,dy = -xy\,dx$$

$$\frac{1}{y}\,dy = -\frac{x}{x+1}\,dx \qquad x+1 \neq 0 \qquad y \neq 0$$

$$x \neq -1$$

$$\ln|y| = \int \frac{-x}{x+1}\,dx = -\int \frac{x}{x+1}\,dx = -\int \left(1 - \frac{1}{x+1}\right)dx = -\int 1\,dx + \int \frac{1}{x+1}\,dx =$$

$$= \left| \begin{matrix} u = x+1 \\ du = dx \end{matrix} \right| = -x + \int \frac{1}{u}\,du = -x + \ln|u| = -x + \ln|x+1| + c$$

$$\ln|y| = -x + \ln|x+1| + c, \quad c \in R$$

$$y = e^{-x + \ln|x+1| + c}$$

$$y = K e^{-x}(x+1), \quad K \in R \setminus \{0\}$$

3) $y - y^2 + xy' = 0$

$$x\frac{dy}{dx} = y^2 - y \qquad y \neq 1,\ y \neq 0,\ x \neq 0$$

$$\frac{1}{y(y-1)}\,dy = \frac{1}{x}\,dx$$

$$\frac{1}{y(y-1)} = \frac{A}{y} + \frac{B}{y-1}$$

$$1 = A(y-1) + By$$

$$A + B = 0$$

$$-A = 1$$

$$A = -1$$

$$-1 + B = 0$$

$$B = 1$$

$$\int \frac{1}{y(y-1)}\,dy = \int \frac{-1\,dy}{y} + \int \frac{1}{y-1}\,dy =$$

$$= -\ln|y| + \ln|y-1| = \ln\left|\frac{y-1}{y}\right|$$

$$\ln\left|\frac{y-1}{y}\right| = \ln|x| + c, \quad c \in R$$

$$\frac{y-1}{y} = Kx, \quad \text{[illegible]} \quad K \in R \setminus \{0\}$$

$$y - 1 = Kyx$$

$$y - Kyx = 1$$

$$y(1 - Kx) = 1$$

$$y = \frac{1}{1-Kx}$$

$y=0$ je řš. a nelze získat žádnou volbou konstanty k $\Rightarrow$ singulární řš.

6) $xy' + y\ln x = y\ln y$ Homogenní rovnice

používáme substituci

$u = \frac{y}{x}$, $y = u\cdot x$, $y' = u'x + u$

$y' = \frac{y}{x}(\ln y - \ln x)$

$y' = \frac{y}{x}\left(\ln \frac{y}{x}\right)$

subst.: $u = \frac{y}{x} \Rightarrow y = u\cdot x \Rightarrow y' = u'x + u$

$u'x + u = u(\ln u)$

$\frac{du}{dx}\cdot x = u\ln u - u$

$\int \frac{1}{u(\ln u - 1)}\,du = \int \frac{1}{x}\,dx$ $u(\ln u - 1) \neq 0$

$u \neq 0$ $\ln u \neq 1$

$u \neq e$

$\int \frac{1}{u(\ln u - 1)}\,du = \left| \begin{array}{l} \ln u - 1 = a \\ \frac{1}{u}\,du = da \end{array} \right| = \int \frac{da}{a} = \ln|a| = \ln|\ln u - 1|$

$\ln|\ln u - 1| = \ln|x| + C$, $C \in \mathbb{R}$

$\ln u - 1 = kx$, $k \in \mathbb{R}\setminus\{0\}$

$\ln u = kx + 1$

$u = e^{kx+1}$

$\frac{y}{x} = e^{kx+1}$

$y = xe^{kx+1}$, $k \in \mathbb{R}\setminus\{0\}$

$u = 0 \Rightarrow \frac{y}{x} = 0 \Rightarrow y = 0$: $0 + 0 = 0$ ✓ po dosazení to sice sedí, ale POZOR! nesplňuje definiční obor rovnice, pač je tam $\ln y \Rightarrow y > 0$.

$u = e \Rightarrow \frac{y}{x} = e \Rightarrow y = ex$: $x\cdot e + ex\ln x = ex\ln ex$

$ex + ex\ln x = ex(\ln e + \ln x)$

$ex(1 + \ln x) = ex(1 + \ln x)$ ✓ ale toto řeš. je možné získat pro $k = 0$, a tedy stačí rozšířit obor k

obecné řeš.: $y = xe^{kx+1}$, $k \in \mathbb{R}$

4) $y' \cos^2 x = (1+\cos^2 x)\sqrt{1-y^2}$

$\frac{dy}{dx} \cos^2 x = (1+\cos^2 x)\sqrt{1-y^2}$ $\qquad 1-y^2 \neq 0$

$\qquad y \neq \pm 1$

$\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{1+\cos^2 x}{\cos^2 x}\, dx$

$\arcsin y = \operatorname{tg} x + x + c$; $c \in R$

$y = \sin(\operatorname{tg} x + x + c)$

$y = 1$: $0 = 0$ ✓ po dosazení do rovnice

$y = -1$: $0 = 0$ ✓ —— // ——

obecné řeš.: $y = \sin(\operatorname{tg} x + x + c)$, $c \in R$

singulární řeš.: $y = \pm 1$ (ověřte, zda je nelze získat nějakou volbou c)

5) $y' \operatorname{tg} x - y^2 = 1 - 2y$

$\frac{dy}{dx} \operatorname{tg} x = y^2 - 2y + 1$ $\qquad y \neq 1$

$\int \frac{1}{y^2-2y+1}\, dy = \int \frac{1}{\operatorname{tg} x}\, dx$

$\int \frac{1}{y^2-2y+1}\, dy = \int \frac{dy}{(1-y)^2} = \left| \begin{matrix} u = y-1 \\ du = dy \end{matrix} \right| = \int \frac{1}{u^2}\, du = -\frac{1}{u} = -\frac{1}{y-1} = \frac{1}{1-y}$

$\int \frac{1}{\operatorname{tg} x}\, dx = \int \frac{\cos x}{\sin x}\, dx = \left| (\sin x)' = \cos x \right| = \ln|\sin x| + c$

$\frac{1}{1-y} = \ln|\sin x| + c$, $c \in R$

$1 - y = \frac{1}{\ln(\sin x) + c}$, $c \in R$

$y = 1 - \frac{1}{\ln(\sin x) + c}$, $c \in R$

$y = 1$: $0 - 1^2 = 1 - 2$

$-1 = -1$ ✓

obecné řeš.: $y = 1 - \frac{1}{\ln(\sin x) + c}$

singulární řeš.: $y = 1$

7) $(xy' - y)\cos\frac{y}{x} = x$

$\left(y' - \frac{y}{x}\right)\cos\frac{y}{x} = 1$

subst.: $u = \frac{y}{x}$, $y' = u'x + u$

$(u'x + u - u)\cos u = 1$

$x\frac{du}{dx}\cos u = 1$

$\int \cos u \, du = \int \frac{1}{x} dx$

$\sin u = \ln|x| + c,\ c \in \mathbb{R}$

$\frac{y}{x} = \arcsin(\ln|x| + c)$

obecné ř.: $y = x \arcsin(\ln|x| + c),\ c \in \mathbb{R}$

Rovnice v podílovém tvaru

vyřešíme jako soustavu rovnic: čitatel$=0$, jmenovatel $=0$

$x = a \quad y = b$

Výsledek nám dá posuny po osách, uděláme substituci $\left.\begin{array}{l} u = x - a \\ v = y - b \end{array}\right\} \Rightarrow$

$du = dx$

$dv = dy$

Po dosazení vyjde homogenní rovnice.

8) $y' = \frac{x + 2y - 7}{x - 3}$

$x + 2y - 7 = 0$

$x - 3 = 0$

$x = 3$

$3 + 2y - 7 = 0$

$2y = 4$

$y = 2$

$u = x - 3 \Rightarrow x = u + 3$

$v = y - 2 \Rightarrow y = v + 2$

$\frac{dv}{du} = \frac{u + 3 + 2(v + 2) - 7}{u + 3 - 3}$

$\frac{dv}{du} = \frac{u + 2v}{u}$

$\frac{dv}{du} = 1 + 2\frac{v}{u}$

$\frac{v}{u} = a \Rightarrow v' = a'u + a$

$a'u + a = 1 + 2a$

$\frac{da}{du} u = 1 + a$

$\frac{1}{a+1} da = \frac{1}{u} du$

$\ln|a + 1| = \ln|u| + c,\ c \in \mathbb{R}$

$\ln\left|\frac{v}{u} + 1\right| = \ln|u| + c$

$1 + \frac{v}{u} = Ku,\ K \in \mathbb{R}\setminus\{0\}$

$v = Ku^2 - u$

$y - 2 = K(x-3)^2 - (x - 3),\ K \in \mathbb{R}\setminus\{0\}$

$a + 1 \neq 0$

$a = -1$

$\frac{v}{u} \neq -1$

$v \neq -u$

$y - 2 \neq -x + 3$

$y \neq -x + 5$

$y = -x + 5$: dostaneme volbou $K = 0$

obecné ř.: $y = K(x-3)^2 - x + 5$

9) $y' = \dfrac{5y-5x-1}{2y-2x-1}$

Zde nejde použít předchozí postup, pzt. jsou u x a y stejné koeficienty.

subst.: $z = y - x \Rightarrow y = z + x \Rightarrow z' + 1 = y'$

$$z' + 1 = \frac{5z-1}{2z-1}$$

$$z' = \frac{5z-1-2z+1}{2z-1}$$

$$\frac{dz}{dx} = \frac{3z}{2z-1}$$

$$\cancel{\frac{dz}{dx}} = \quad \int \frac{2z-1}{3z}\,dz = \int dx \qquad z \neq 0 \rightarrow y - x \neq 0 \Rightarrow y \neq x$$

$$\frac{2}{3}\int dz - \frac{1}{3}\int \frac{1}{z}\,dz = x + c, \quad c \in \mathbb{R}$$

$$\frac{2}{3}z - \frac{1}{3}\ln|z| = x + c$$

$$\frac{2}{3}(y-x) - \frac{1}{3}\ln|y-x| = x + c$$

$$2y - 2x - \ln|y-x| = 3x + k, \quad k \in \mathbb{R}$$

$y = x$: $0 = \dfrac{5x-5x-1}{2x-2x-1}$ ✓, ale to by nevyhovovalo definičnímu oboru rovnice

10) $2(1+e^x)yy' = e^x \qquad y(0) = 0$

$$2(1+e^x)y\frac{dy}{dx} = e^x$$

počáteční podmínka: slouží k určení partikulárního řeš.

$$\int y\,dy = \int \frac{e^x}{2(1+e^x)}\,dx$$

partikulární řeš.: řeš. po dosazení konkrétní hodnoty za integrační konstantu, tuto hodnotu konstanty získáme dosazení počáteční podmínky do obecného řeš.

$$\frac{y^2}{2} = \frac{1}{2}\ln|1+e^x| + c, \quad c \in \mathbb{R}$$

$$y^2 = \ln(1+e^x) + c$$

$$y = \pm\sqrt{\ln(1+e^x) + c}$$

$y(0) = 0 \Rightarrow$ za x dosadím 0, za y dosadím 0

$$0 = \sqrt{\ln(1+1) + c}$$

$$c = -\ln 2$$

$$y_p = \pm\sqrt{\ln(1+e^x) - \ln 2}$$

partikulární řeš.

11) $\sin y \cos x \, dy = \cos y \sin x \, dx \qquad y(0) = \frac{\pi}{4}$

$\operatorname{tg} y \, dy = \operatorname{tg} x \, dx$

$-\ln|\cos y| = -\ln|\cos x| + C, \ C \in \mathbb{R}$

$\cos y = k \cdot \cos x, \ k \in \mathbb{R}$

$y = \arccos(k \cos x)$

$\frac{\pi}{4} = \arccos(k \cdot \cos 0)$

$\frac{\pi}{4} = \arccos k$

$k = \cos \frac{\pi}{4}$

$k = \frac{\sqrt{2}}{2}$

$y_p = \arccos\left(\frac{\sqrt{2}}{2} \cos x\right)$

12) $(x^2+1)(y^2-1) + xyy' = 0 \qquad y(1) = \sqrt{2}$

$(x^2+1)(y^2-1) + xy\frac{dy}{dx} = 0$

$xy\frac{dy}{dx} = (x^2+1)(-y^2+1)$

$\int \frac{y}{1-y^2} dy = \int \frac{x^2+1}{x} dx$

$-\frac{1}{2}\int \frac{-2y}{1-y^2} dy = \int \left(x + \frac{1}{x}\right) dx$

$-\frac{1}{2}\ln|1-y^2| = \frac{x^2}{2} + \ln|x| + C \quad C \in \mathbb{R}$

$\ln|1-y^2| = -x^2 - 2\ln|x| + k_1, \ k_1 \in \mathbb{R}$

$1 - y^2 = e^{-x^2} \cdot x^{-2} \cdot k, \ k \in \mathbb{R}$

$y^2 = 1 + ke^{-x^2} x^{-2}$

$y = \pm\sqrt{1 + ke^{-x^2} x^{-2}}$

$2 = 1 + ke^{-1} \cdot 1$

$k = e$

$y_p = \pm\sqrt{1 + e^{-x^2+1} x^{-2}}$

Lineární rovnice $y' = f(x,y)$

13) $y' = 6x - 2y$

a) metoda variace konstant – zanedbáme x na pravé straně, tím dostaneme homogenní část lineární diferenciální rovnice

$$y' = -2y$$

$$\frac{dy}{dx} = -2y \qquad y \neq 0$$

$$\int \frac{1}{y}\,dy = \int -2\,dx$$

$$\ln|y| = -2x + C, \; C \in \mathbb{R}$$

$$|y| = e^{-2x} \cdot K, \; K > 0$$

$$y = Le^{-2x}; \; L \in \mathbb{R} \setminus \{0\}$$

$y = 0: \; 0 = -2 \cdot 0$ ✓ $\Rightarrow L \in \mathbb{R}$

řeš. homogenní rovnice $y_H = Le^{-2x}, \; L \in \mathbb{R}$

Nyní budeme uvažovat, že L není jen reálné číslo, ale nějaká f-ce závislá na x.

$$y = \underbrace{L(x)e^{-2x}}$$ a budeme počítat derivaci

$$y' = \underbrace{L'(x) \cdot e^{-2x} + L(x) \cdot e^{-2x}(-2)},$$ pak dosadíme do původní rovnice za y a y'.

$$L'(x)e^{-2x} \cancel{- 2L(x)e^{-2x}} = 6x \cancel{- 2Le^{-2x}}$$

Všechny členy, kde se vyskytuje $L(x)$ se musí vzájemně odečíst, jinak je někde chyba.

$$L'(x)e^{-2x} = 6x$$

$$L'(x) = e^{2x}6x$$

$$L(x) = \int e^{+2x}6x\,dx = \left| \begin{matrix} u = 6x & v' = e^{2x} \\ u' = 6 & v = \frac{1}{2}e^{2x} \end{matrix} \right| = 3xe^{2x} - \int 3e^{2x}dx = 3xe^{2x} - \frac{3}{2}e^{2x} + C$$

Nyní za konstantu dosadíme do y_H a dostaneme obecné řeš.

$$y = 3x - \frac{3}{2} + Ce^{-2x}, \; C \in \mathbb{R}$$

2) $y' = 4xy + (2x+1)e^{2x^2}$

$$y' - 4xy = (2x+1)e^{2x^2} \quad / \cdot e^{-\int 4x\,dx} = e^{-2x^2}$$

$$\left(ye^{-2x^2}\right)' = (2x+1)e^{2x^2}\cdot e^{-2x^2}$$

$$ye^{-2x^2} = 2\cdot\frac{x^2}{2} + x + C$$

Ob. řeš.: $y = x^2e^{2x^2} + xe^{2x^2} + Ce^{2x^2}$, $C \in \mathbb{R}$

3) $y'\cos x = (y + 2\cos x)\sin x$

$$y' = (y + 2\cos x)\,\mathrm{tg}\,x$$

homogenní: $y' = y\,\mathrm{tg}\,x$

$$\frac{dy}{dx} = y\,\mathrm{tg}\,x$$

$$\frac{dy}{y} = \mathrm{tg}\,x\,dx$$

$$\ln|y| = -\ln|\cos x| + C, \quad C \in \mathbb{R}$$

$$y_H = \frac{k}{\cos x}, \quad k \in \mathbb{R}\setminus\{0\}$$

$$y' = \frac{k'(x)\cos x + k(x)\sin x}{\cos^2 x}$$

$$\frac{k'(x)\cos x + k(x)\sin x}{\cos^2 x}\cdot\cos x = \left(\frac{k(x)}{\cos x} + 2\cos x\right)\sin x$$

$$k'(x) = 2\cos x\sin x$$

$$k(x) = \int 2\cos x\sin x = \begin{vmatrix} t = \sin x \\ dt = \cos x\,dx \end{vmatrix} = 2\int t\,dt = 2\frac{t^2}{2} =$$

$$= \sin^2 x + C$$

Ob. řeš.: $y = \dfrac{\sin^2 x + C}{\cos x}$ $\quad C \in \mathbb{R}$

4) $y' - 4y = \cos x \;/\cdot e^{\int -4\,dx}$ $\quad y(0) = 1$

$$\left(e^{-4x}y\right)' = e^{-4x}\cos x$$

$$ye^{-4x} = \int e^{-4x}\cos x\,dx$$

$$\int e^{-4x}\cos x\,dx = \begin{vmatrix} u = \cos x & v' = e^{-4x} \\ u' = -\sin x & v = -\frac{1}{4}e^{-4x} \end{vmatrix} = -\frac{1}{4}e^{-4x}\cos x -$$

$$- \int \frac{1}{4}e^{-4x}\sin x\,dx = \begin{vmatrix} u = \frac{1}{4}\sin x & v' = e^{-4x} \\ u' = \frac{1}{4}\cos x & v = -\frac{1}{4}e^{-4x} \end{vmatrix} =$$

$$= -\frac{1}{4}e^{-4x}\cos x - \left(-\frac{1}{16}e^{-4x}\sin x + \int\frac{1}{16}e^{-4x}\cos x\,dx\right) =$$

$$= -\frac{1}{4}e^{-4x}\cos x + \frac{1}{16}e^{-4x}\sin x - \frac{1}{16}\int e^{-4x}\cos x\,dx$$

$$\frac{17}{16}\int e^{-4x}\cos x\,dx = -\frac{1}{4}e^{-4x}\cos x + \frac{1}{16}e^{-4x}\sin x$$

$\int e^{-4x}\cos x\,dx = -\frac{4}{17}e^{-4x}\cos x + \frac{1}{17}e^{-4x}\sin x + C$

$y e^{-4x} = -\frac{4}{17}e^{-4x}\cos x + \frac{1}{17}e^{-4x}\sin x + C$

ob. řeš.: $y = -\frac{4}{17}\cos x + \frac{1}{17}\sin x + Ce^{4x}$

$1 = -\frac{4}{17}\cos 0 + \frac{1}{17}\sin 0 + Ce^{4\cdot 0}$

$1 = -\frac{4}{17} + C$

$C = \frac{21}{17}$

$y_p = -\frac{4}{17}\cos x + \frac{1}{17}\sin x + \frac{21}{17}e^{4x}$

Bernoulliova rovnice

$y' = a(x)y + b(x)y^p,\ p\in\mathbb{R},\ p\neq 0,\ p\neq 1$

$z = \frac{1}{y^{p-1}} \Rightarrow z' = (1-p)\frac{1}{y^p}\cdot y'$

Tato substituce převede Bernoulliovu rovnici na lineární.

5) $3x^2y' + xy = y^{-2}$

$y' = -\frac{y}{3x} + \frac{1}{3x^2}\cdot y^{-2} \quad /\cdot 3y^2$

$z = \frac{1}{y^{-3}} = y^3 \Rightarrow z' = 3y^2\cdot y'$

$3y^2y' = -\frac{y^3}{x} + \frac{1}{x^2}$

$z' = -\frac{1}{x}z + \frac{1}{x^2}$

$z' + z\frac{1}{x} = \frac{1}{x^2} \quad / \quad e^{\int\frac{1}{x}dx} = e^{\ln|x|} = \pm x$

$z'x + z = \frac{1}{x}$ $\qquad -z'x - z = -\frac{1}{x}$

$\qquad z'x + z = \frac{1}{x}$

$(xz)' = \frac{1}{x}$

$xz = \ln|x| + c$

$z = \frac{\ln|x| + c}{x}$

$y^3 = \frac{\ln|x| + c}{x},\ C\in\mathbb{R}$

6, $y' = \frac{4}{x} y + x\sqrt{y}$

$z = \sqrt{y} \qquad z' = \frac{1}{2\sqrt{y}} \cdot y'$

$\frac{1}{2\sqrt{y}} y' = \frac{2}{x}\sqrt{y} + \frac{x}{2} \qquad y \neq 0$

$z' = \frac{2}{x} z + \frac{x}{2}$

$z' - \frac{2}{x} z = \frac{x}{2} \quad \Big/ e^{-2\int \frac{1}{x} dx} = e^{-2\ln|x|} = x^{-2} = \frac{1}{x^2}$

$\left(\frac{z}{x^2}\right)' = \frac{1}{2x}$

$\frac{z}{x^2} = \frac{1}{2}\ln|x| + C$

$\frac{\sqrt{y}}{x^2} = \frac{1}{2}\ln|x| + C$

$y = \left(\frac{1}{2}x^2 \ln|x| + Cx^2\right)^2, \; C \in \mathbb{R}$

$y = 0: \quad 0 = \frac{4}{x}\cdot 0 + x\sqrt{0}$ ✓

ob. ř.: $y = \left(\frac{1}{2}x^2\ln|x| + Cx^2\right)^2, \; C \in \mathbb{R}, \quad y = 0$

7, $xy' + 2y + x^5 y^3 e^x = 0$

$y' = -\frac{2y}{x} - e^x x^4 y^3 \quad \Big/ \cdot \frac{-2}{y^3}$

$z = \frac{1}{y^{3-1}} = \frac{1}{y^2} \Rightarrow z' = -2\frac{1}{y^3} y'$

$-\frac{2}{y^3} y' = \frac{4}{xy^2} + 2e^x x^4$

$z' = \frac{4}{x} z + 2e^x x^4$

$z' - \frac{4}{x} z = 2e^x x^4 \quad \Big/ e^{-\int \frac{4}{x} dx} = e^{-4\ln|x|} = \frac{1}{x^4}$

$\left(\frac{z}{x^4}\right)' = 2e^x$

$\frac{z}{x^4} = 2e^x + C$

$\frac{1}{y^2 x^4} = 2e^x + C$

$\frac{1}{y^2} = 2x^4 e^x + Cx^4, \; C \in \mathbb{R}$

$y = 0: \quad x\cdot 0 + 2\cdot 0 + x^5 0^3 e^x = 0$ ✓

ob. ř.: $\frac{1}{y^2} = 2x^4 e^x + Cx^4, \; C \in \mathbb{R}, \; y = 0$

8. $y'x + y = x\ln x$

$y' + \frac{y}{x} = \ln|x| \quad / \cdot e^{\int \frac{1}{x} dx} = e^{\ln|x|} = x$

$(yx)' = x\ln|x|$

$\int x\ln x\, dx = \left| \begin{matrix} u = \ln x & v' = x \\ u' = \frac{1}{x} & v = \frac{x^2}{2} \end{matrix} \right| = \frac{x^2}{2}\ln x - \int \frac{x}{2} dx =$

$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$

$yx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$

$y = \frac{x}{2}\ln x - \frac{x}{4} + \frac{C}{x}, \; C \in \mathbb{R}$

9. $y' + y\sin x = \sin x \qquad y\left(\frac{\pi}{2}\right) = 2$

$y' + y\sin x = \sin x \quad / e^{\int \sin x dx} = e^{-\cos x}$

$(ye^{-\cos x})' = \sin x e^{-\cos x}$

$ye^{-\cos x} = \int \sin x\, e^{-\cos x} dx$

$\int \sin x\, e^{-\cos x} dx = \left| \begin{matrix} t = -\cos x \\ dt = \sin x dx \end{matrix} \right| = \int e^t dt = e^t = e^{-\cos x} + C$

$ye^{-\cos x} = e^{-\cos x} + C$

$y = 1 + Ce^{\cos x}$

$2 = 1 + Ce^{\cos \frac{\pi}{2}}$

$1 = Ce^0$

$C = 1$

$y_p = 1 + e^{\cos x}$

Záměna proměnných

$y' = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{x'}$

10. $y' = \frac{1}{2x - y^2}$

$\frac{1}{x'} = \frac{1}{2x - y^2}$

$x' = 2x - y^2$

$x' - 2x = -y^2 \quad / e^{\int (-2) dy} = e^{-2y}$

$$(xe^{-2y})' = -e^{-2y}y^2$$

$$\int y^2 e^{-2y}\,dy = \left| \begin{matrix} u = y^2 & v' = e^{-2y} \\ u' = 2y & v = -\frac{1}{2}e^{-2y} \end{matrix} \right| = -\frac{1}{2}y^2e^{-2y} +$$

$$+ \int y e^{-2y}\,dy = \left| \begin{matrix} u = y & v' = e^{-2y} \\ u' = 1 & v = -\frac{1}{2}e^{-2y} \end{matrix} \right| = -\frac{1}{2}y^2e^{-2y} -$$

$$-\frac{1}{2}ye^{-2y} + \frac{1}{2}\int e^{-2y}\,dy = -\frac{1}{2}y^2e^{-2y} - \frac{1}{2}ye^{-2y} + \frac{1}{2}\left(-\frac{1}{2}\right)e^{-2y} + C$$

$$xe^{-2y} = +\frac{1}{2}y^2e^{-2y} + \frac{1}{2}ye^{-2y} + \frac{1}{4}e^{-2y} + C$$

$$x = +\frac{1}{2}y^2 + \frac{1}{2}y + \frac{1}{4} + Ce^{2y}, \quad C \in R$$

11) $$y' = \frac{y}{2y\ln y + y - x}$$

$$x' = \frac{2y\ln y + y - x}{y}$$

$$x' = 2\ln y + 1 - \frac{x}{y}$$

$$x' + \frac{x}{y} = 2\ln y + 1 \quad \Big/ e^{\int \frac{1}{y}dy} = y$$

$$(xy)' = 2\int y\ln y + \int y\,dy$$

$$\int y\ln y\,dy = \left| \begin{matrix} u = \ln y & v' = y \\ u' = \frac{1}{y} & v = \frac{y^2}{2} \end{matrix} \right| = \frac{y^2}{2}\ln y - \frac{1}{2}\int y\,dy =$$

$$= \frac{y^2}{2}\ln y - \frac{1}{2}\cdot\frac{y^2}{2} + \frac{y^2}{2}$$

$$xy = y^2\ln y - \frac{y^2}{2} + \frac{y^2}{2} + C$$

$$x = y\ln y + \frac{C}{y}, \quad C \in R$$

CVIČENÍ 3

Exaktní dif. rovnice

$$P(x,y)\,dx + Q(x,y)\,dy = 0$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Řeš. hledáme ve tvaru $F(x,y)=C$ ... kmenová f-e, pokud $\frac{\partial F}{\partial x}(x,y) = P(x,y)$ a $\frac{\partial F}{\partial y} = Q(x,y)$

1) $(x\cos 2y + 1)\,dx - x^2 \sin 2y\,dy = 0$

$\frac{\partial P}{\partial y} = -2x \sin 2y \qquad \frac{\partial Q}{\partial x} = -2x \sin 2y$

$F(x,y) = -\int x^2 \sin 2y\,dy + C(x) = -x^2\int \sin 2y\,dy = \frac{1}{2}x^2 \cos 2y + C(x)$

$\frac{\partial F}{\partial x} = x\cos 2y + 1$

$F_x = \left(\frac{1}{2}x^2 \cos 2y + C(x)\right)'_x = \frac{1}{2}\cos 2y \cdot 2x + C'(x)$

$x\cos 2y + C'(x) = x \cos 2y + 1$

$C'(x) = 1$

$C(x) = x + C \quad \Rightarrow F(x,y) = \frac{1}{2}x^2 \cos 2y + x + C$

ob. řeš.: $C = \frac{1}{2}x^2\cos 2y + x,\ C \in \mathbb{R}$

$y = \arccos \frac{2C - 2x}{x^2},\ C \in \mathbb{R}$

8. 3. 2011

2) $(e^y + ye^x + 3x^2)\,dx = (2 - xe^y - e^x)\,dy$

$\frac{\partial P}{\partial y} = e^y + 1e^x \qquad \frac{\partial Q}{\partial x} = \cancel{x}e^y + e^x$

$F(x,y) = \int(-2 + xe^y + e^x)\,dy + C(x) = -2y + xe^y + ye^x + C(x)$

$+\cancel{x}e^y + ye^x + C'(x) = e^y + ye^x + 3x^2$

$C'(x) = 3x^2$

$C(x) = \frac{3}{3}x^3 + C \quad \Rightarrow F(x,y) = -2y + xe^y + ye^x + x^3 + C$

ob. řeš.: $C = -2y + xe^y + ye^x + x^3,\ C \in \mathbb{R}$

3) $\sin y\,dx + [(x+1)\cos y - y \sin y]\,dy = 0$

$\frac{\partial P}{\partial y} = \cos y \qquad \frac{\partial Q}{\partial x} = \cos y$

$F(x,y) = \int \sin y\,dx + C(y) = x \sin y + C(y)$

$x\cos y + C'(y) = [(x+1)\cos y - y\sin y]$

$C'(y) = \cos y - y\sin y$

$C(y) = \sin y - \int y\sin y\,dy = \begin{vmatrix} u = y & v' = -\sin y \\ u' = 1 & v = \cos y \end{vmatrix} =$

$= \sin y + y\cos y - \int \cos y\,dy = \sin y + y\cos y - \sin y = y\cos y$

ob. řeš.: $C = x\sin y + y\cos y$, $C \in \mathbb{R}$

4) $\left(\frac{1}{y^2+1} - \frac{y}{x^2}\right)dx + \left(\frac{1}{x} - \frac{2xy}{(y^2+1)^2}\right)dy = 0$

$\frac{\partial P}{\partial y} = -\frac{2y}{(y^2+1)^2} - \frac{1}{x^2}$ $\qquad \frac{\partial Q}{\partial x} = -\frac{1}{x^2} - \frac{2y}{(y^2+1)^2}$

$F(x,y) = \int\left(\frac{1}{y^2+1} - \frac{y}{x^2}\right)dx + C(y) = \frac{x}{y^2+1} + \frac{y}{x} + C(y)$

$-\frac{x\,2y}{(y^2+1)^2} + \frac{1}{x} + C'(y) = \frac{1}{x} - \frac{2xy}{(y^2+1)^2}$

$C'(y) = 0 \Rightarrow C(y) = C$, $C \in \mathbb{R}$

ob. řeš.: $C = \frac{x}{y^2+1} + \frac{y}{x}$, $C \in \mathbb{R}$

Pokud rovnice není exaktní, pokusíme se najít integrační faktor, kterým upravíme rovnici vynásobením, aby vzniklá rovnice exaktní.

Integrační faktor může být tvaru

$\ln m(x) = \int \frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q}\,dx$ nebo

$\ln m(y) = \int \frac{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}{P}\,dy$

5) $(x^2 - 3y^2)dx + 2xy\,dy = 0$

$\frac{\partial P}{\partial y} = -6y$ $\qquad \frac{\partial Q}{\partial x} = 2y$. $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x} \Rightarrow$ není exaktní

$\ln m(x) = \int \frac{-6y - 2y}{2xy}\,dx = \int \frac{-4}{x}\,dx = -4\ln|x|$

$m(x) = \frac{1}{x^4}$

$\left(\frac{1}{x^2} - \frac{3y^2}{x^4}\right)dx + \frac{2y}{x^3}\,dy = 0$

$\frac{\partial P}{\partial y} = -\frac{6y}{x^4}$ $\qquad \frac{\partial Q}{\partial x} = -\frac{3\cdot 2y}{x^4}$

$F(x,y) = \int \left(\frac{2y}{x^3}\right)dy + C(x) = \frac{y^2}{x^3} + C(x)$

$$-\frac{3y^2}{x^4} + C'(x) = \frac{1}{x^2} - \frac{3y^2}{x^4}$$

$$C'(x) = \frac{1}{x^2}$$

$$C(x) = -\frac{1}{x}$$

Op. rj.: $C = \frac{y^2}{x^3} - \frac{1}{x}$, $C \in \mathbb{R}$

6) $2xy \ln y \, dx + (x^2 + y^2\sqrt{y^2+1})\, dy = 0$

$\frac{\partial P}{\partial y} = 2x \ln y + 2xy \cdot \frac{1}{y}$ $\quad \frac{\partial Q}{\partial x} = 2x \Rightarrow$ [illegible]

$$\ln m(y) = \int \frac{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}{P}\, dy = \int \frac{2x - 2x\ln y - 2x}{2xy \ln y}\, dy =$$

$$= \int -\frac{1}{y}\, dy = -\ln|y|$$

$$\ln m(y) = -\ln|y|$$

$$m(y) = \frac{1}{y}$$

$$2x \ln y \, dx + \left(\frac{x^2}{y} + y\sqrt{y^2+1}\right) dy = 0$$

$\frac{\partial P}{\partial y} = \frac{2x}{y}$ $\quad \frac{\partial Q}{\partial x} = \frac{2x}{y}$

$$F(x,y) = 2\int x \ln y \, dx + C(y) = 2\frac{x^2}{2}\ln y + C(y) = x^2 \ln y + C(y)$$

$$\frac{x^2}{y} + C'(y) = \frac{x^2}{y} + y\sqrt{y^2+1}$$

$$C'(y) = y\sqrt{y^2+1}$$

$$C(y) = \int y\sqrt{y^2+1}\, dy = \left| \begin{matrix} y^2+1 = t \\ 2y\, dy = dt \end{matrix} \right| = \frac{1}{2}\int \sqrt{t}\, dt =$$

$$= \frac{1}{2}\int t^{\frac{1}{2}}\, dt = \frac{1}{2} \cdot t^{\frac{3}{2}} \cdot \frac{2}{3} = \frac{1}{3}\sqrt{t^3} = \frac{1}{3}\sqrt{(y^2+1)^3}$$

Op. rj.: $C = x^2 \ln y + \frac{1}{3}\sqrt{(y^2+1)^3}$, $C \in \mathbb{R}$

Clairautova rovnice

$$y = xy' + g(y')$$

1. zavedeme substituci $y' = p$
2. celou rovnici zderivujeme

y, p považujeme jako funkce s parametrem x v konečné fázi: nesmíme zapomenout, že závisí na x!

7) $y = xy' + (y')^2$

subst.: $y' = p$

$$\boxed{y = xp + p^2} \quad / \frac{d}{dx}$$

$$y' = p + x \cdot p' + 2pp'$$

$$p = p + xp' + 2pp'$$

$$0 = p'(x + 2p)$$

$p' = 0$ | $x + 2p = 0$
$\frac{dp}{dx} = 0$ | $-2p = x \Rightarrow p = -\frac{1}{2}x$
$p = C$ | $y = -2p \cdot p + p^2 = -p^2$
$\underline{y = x \cdot C + C^2, \ C \in \mathbb{R}}$ | $\underline{y = -\frac{1}{4}x^2}$

8) $y = xy' + \sin y'$?

$$y = xp + \sin p \quad / \frac{d}{dx}$$

$$p = p + xp' + \cos p \cdot p'$$

$$0 = (x + \cos p)p'$$

$\frac{dp}{dx} = 0$

$p = C$

$y = x \cdot C + \sin C, \ C \in \mathbb{R}$

$x = -\cos p \Rightarrow p = \arccos(-x) \Rightarrow p = \pi - \arccos x$

~~$y = x(\pi - \arccos x) + \sin(\pi - \arccos x)$~~

$\arcsin x = \arccos \sqrt{1-x^2}$

$\sin(x-y) = \sin x \cos y - \sin y \cos x$

$y = -\cos p \cdot p + \sin p$

$$y = -(\pi - \arccos x)\cos(\pi - \arccos x) + \sin(\pi - \arccos x)$$

$$y = -(\pi - \arccos x)\cdot\cos(\arccos(-x)) + \sin\pi\cos\arccos x - \sin\arccos x\cos\pi$$

$$y = x(\pi - \arccos x) + \sin\arcsin\sqrt{1-x^2}$$

$$y = x(\pi - \arccos x) + \sqrt{1-x^2}$$

9. $xy' - y = \ln y'$

$$xp - y = \ln p$$

$$y = xp - \ln p$$

$$p = p + xp' - \frac{1}{p}\cdot p'$$

$$0 = \left(x - \frac{1}{p}\right)p'$$

$p = C$ $\qquad$ $x = \frac{1}{p} \Rightarrow p = \frac{1}{x}$

$\underline{y = Cx - \ln C,\ C>0}$ $\qquad$ $y = \frac{1}{p}\cdot p - \ln p$

$$y = 1 - \ln p$$

$$y = 1 - \ln\frac{1}{x}$$

$$\underline{y = 1 + \ln x}$$

10. $y = xy' + \frac{1}{2y'}$

$$y = xp + \frac{1}{2p}$$

$$p = y' = xp' + p - \frac{2p'}{2p^2}$$

$$p = xp' + p - \frac{1\cdot p'}{2p^2}$$

$$0 = xp' - \frac{1}{2p^2}p'$$

$$0 = \left(x - \frac{1}{2p^2}\right)p'$$

$p = C$ $\qquad$ $x = \frac{1}{2p^2} \Rightarrow 2p^2 = \frac{1}{x} \Rightarrow p = \pm\sqrt{\frac{1}{2x}}$

$y = C\cdot x + \frac{1}{2C},\ C\in\mathbb{R}\setminus\{0\}$ $\qquad$ $y = \frac{1}{2p^2}\cdot p + \frac{1}{2p}$

$$y = \frac{1}{2p} + \frac{1}{2p} = \frac{1}{p}$$

$$y^2 = 2x$$

11) $y = xy' + y' + e^{y'}$

$y = xp + p + e^{p}$

$p = p + xp' + p' + e^{p} \cdot p'$

$0 = p'(x + 1 + e^{p})$

$p' = 0$

$p = C$

$y = Cx + C + e^{C}, \; C \in R$

$\rightarrow x = -1 - e^{p} \Rightarrow e^{p} = -1 - x$

$p = \ln(-1-x)$

$y = (-1-e^{p})p + p + e^{p}$

$y = -1p - e^{p}p + p + e^{p}$

$y = -e^{p}p + e^{p}$

$y = (1-p)e^{p}$

$y = [1 - \ln(-1-x)] \cdot e^{\ln(-1-x)}$

$y = (-1-x)(1 - \ln(-1-x))$

$y = -1 - x + \ln(-1-x) + x\ln(-1-x)$

$y = (-1-x) + \ln(-1-x)(1+x)$

$y = (-1 + \ln(-1-x))(x+1)$

12) $\left(\frac{6y}{x} - 6y^2\right)dx + (3 - 4xy)\,dy = 0$

$\frac{\partial P}{\partial y} = \frac{6}{x} - 12y \qquad \frac{\partial Q}{\partial x} = -4y \rightarrow$ není exaktní

$\ln m(x) = \int \frac{\frac{6}{x} - 12y + 4y}{3-4xy}\,dx = \int \frac{6 - 12xy + 4xy}{x} \cdot \frac{1}{3-4xy}\,dx =$

$= \int \frac{6 - 8xy}{x} \cdot \frac{1}{3-4xy}\,dx = 2\int \frac{3-4xy}{x} \cdot \frac{1}{3-4xy}\,dx =$

$= 2\int \frac{1}{x}\,dx = 2\ln|x| = \ln|x^2|$

$\ln m(x) = \ln x^2$

$m(x) = x^2$

$(6xy - 6x^2y^2)\,dx + (3x^2 - 4x^3y)\,dy = 0$

$\frac{\partial P}{\partial y} = 6x - 12yx^2 \qquad \frac{\partial Q}{\partial x} = 6x - 12x^2y$

$\cancel{F(x,y) = \int (6xy + 6x^2y^2)\,dx + C(y) = 6x\frac{y^2}{2} + 6x^2\frac{y^3}{3} + C(y)} =$

$= \cancel{3xy^2 + 2x^2y^3 + C(y)}$

$\cancel{\text{[illegible]} \quad 3x \cdot 2y + 2x^2 \cdot 3y^2 + C'(y) = 3x^2 + 4x^3y}$

$F(x,y) = \int (6xy - 6x^2y^2)\,dx = 3x^2y - 2x^3y^2 + C(y)$

$3x^2 - 4x^3y + C'(y) = 3x^2 - 4x^3y$

$C(y) = C, \; C \in \mathbb{R}$

$C = 3x^2y - 2x^3y^2, \; \underline{\underline{C \in \mathbb{R}}}$

Lagrangeova rovnice (d'Alembertova)

$$y = f(y')x + g(y')$$

$$y' = p$$

$$y = f(p)x + g(p) \quad \Big| \frac{d}{dx}$$

$$p = f(p) + [xf'(p) + g'(p)]\frac{dp}{dx}$$

$$p - f(p) = [xf'(p) + g'(p)]\frac{dp}{dx}$$

Pokud $x \neq f(p)$, pak $\dfrac{dx}{dp} = \dfrac{f'(p)}{p-f(p)}x + \dfrac{g'(p)}{p-f(p)}$

1) $y = x(1+y') + (y')^2$.

subst. $y' = p$

$$y = x(1+p) + p^2 \quad \Big| \frac{d}{dx}$$

$$p = 1(1+p) + xp' + 2pp'$$

$$-1 = p'(x+2p)$$

$$-1 = \frac{dp}{dx}(x+2p)$$

$$\frac{1}{-x-2p} = \frac{dp}{dx}$$

$$-x-2p = \frac{dx}{dp}$$

$$\frac{dx}{dp} + x = -2p \quad \Big| e^{\int 1\,dp} = e^p$$

$$(xe^p)' = -2pe^p$$

$$\int(-2pe^p)dp = \begin{vmatrix} u=-2p & v'=e^p \\ u'=-2 & v=e^p \end{vmatrix} = -2pe^p + 2\int e^p dp =$$

$$= -2pe^p + 2e^p + C$$

$$xe^p = 2e^p - 2pe^p + C$$

$$x = 2 - 2p + Ce^{-p}, \quad C \in \mathbb{R}$$

$$y = (2-2p+Ce^{-p})(1+p) + p^2 = 2 + 2p - 2p - 2p^2 +$$

$$+ Ce^{-p} + Cpe^{-p} + p^2 = \underline{\underline{2 - p^2 + Ce^{-p}(1+p)}}, \quad C \in \mathbb{R}$$

2, $y = 2xy' + \ln y'$.

$y' = p$

$y = 2xp + \ln p \quad \Big| \frac{d}{dx}$

$p = 2p + 2xp' + \frac{1}{p}\cdot p' \qquad p \neq 0$

$-p = \frac{dp}{dx}\left(2x + \frac{1}{p}\right)$

$-p\frac{dx}{dp} = 2x + \frac{1}{p} \quad \Big| :(-p)$

$x' = -\frac{2x}{p} - \frac{1}{p^2}$

$x' + \frac{2}{p}x = -\frac{1}{p^2} \quad \Big| \; e^{\int \frac{2}{p}dp} = e^{2\ln|p|} = p^2$

$(xp^2)' = -1$ $\qquad$ $p=0: \; y = 2x\cdot 0 + \ln 0$

$xp^2 = -p + C$ $\qquad$ neodpovídá def. oboru!

$x = -\frac{1}{p} + Cp^{-2}, \; C \in R$

$y = 2\left(-\frac{1}{p} + Cp^{-2}\right)p + \ln p$

$\underline{y = -2 + \frac{2C}{p} + \ln p, \; C \in R}$

3, $y - (y')^2(x+1) = 0$. Ú

$y' = p$

$y = p^2(x+1) \quad \Big| \frac{d}{dx}$

$p = 2pp'(x+1) + p^2$

$p = 2pp'x + 2pp' + p^2$

$p - p^2 = p'(2px + 2p)$

$p(1-p) = \frac{dp}{dx}(2x+2)p \qquad p \neq 0 \qquad p=0: \; y' = 0 \Rightarrow \underline{y=0}$ ✓

$1-p = \frac{dp}{dx}(2x+2)$

$(1-p)\frac{dx}{dp} = 2x+2$

$\frac{dx}{dp} = \frac{2x+2}{1-p}$

$x' = \frac{2x}{1-p} + \frac{2}{1-p}$

$x' + \frac{2}{p-1}x = \frac{2}{1-p} \quad \Big| \; e^{\int \frac{2}{p-1}dp} = e^{2\ln|p-1|} = (p-1)^2$

$[x\cdot(p-1)^2]' = \frac{2(p-1)^2}{1-p}$

$[x(p-1)^2]' = \frac{-2(p-1)^2}{p-1}$ $\qquad$ $2(1-p)$

$$\int 2(1-p)\,dp = 2\int 1\,dp - 2\int p\,dp = 2p - p^2 + C$$

$$x(p-1)^2 = 2p - p^2 + C$$

$$x = \frac{2p-p^2+C}{(p-1)^2}, \quad C\in\mathbb{R}$$

$$x = \frac{-p^2+2p-1+C}{(p-1)^2} = \frac{-(p-1)^2+C}{(p-1)^2} = \frac{C}{(p-1)^2} - 1, \quad C\in\mathbb{R}$$

$$y = p^2\left(\frac{C}{(p-1)^2} - 1 + 1\right)$$

$$\underline{y = \frac{Cp^2}{(p-1)^2}, \quad C\in\mathbb{R}} \qquad \underline{y=0}$$

4) $yy' = 2x(y')^2 + 1$

$$y' = p$$

$p \neq 0$ $\qquad p=0:\ y'=0:\ y\cdot 0 = 2x\cdot 0 + 1$ ✗

$$yp = 2xp^2 + 1$$

$$y = 2xp + \frac{1}{p} \quad \Big| \frac{d}{dx}$$

$$p = 2p + 2xp' + (-1)\cdot\frac{1}{p^2}p'$$

$$-p = 2xp' - \frac{p'}{p^2}$$

$$-p = p'\left(2x - \frac{1}{p^2}\right)$$

$$-p\frac{dx}{dp} = \left(2x - \frac{1}{p^2}\right) \quad \Big| : (-p)$$

$$x' = -\frac{2x}{p} + \frac{1}{p^3}$$

$$x' + \frac{2}{p}x = \frac{1}{p^3} \quad \Big| \cdot e^{\int \frac{2}{p}dp} = p^2$$

$$(xp^2)' = \frac{1}{p}$$

$$xp^2 = \ln|p| + C$$

$$x = \frac{\ln|p|+C}{p^2}, \quad C\in\mathbb{R}$$

$$y = 2\left(\frac{\ln|p|+C}{p^2}\right)\cdot p + \frac{1}{p}$$

$$\underline{y = \frac{2}{p}(\ln|p| + C) + \frac{1}{p}}, \quad C\in\mathbb{R}$$

Rovnice vyšších řádů

Lineární dif. r-ce n-tého řádu s konstantními koeficienty $y^{(n)}+a_1y^{(n-1)}+\dots+a_{n-1}y'+a_ny=f(x)$

I, homogenní

$$y^{(n)}+a_1y^{(n-1)}+a_2y^{(n-2)}+\dots+a_{n-1}y'+a_ny=0$$

$$\lambda^n+a_1\lambda^{n-1}+a_2\lambda^{n-2}+\dots+a_{n-1}\lambda+a_n=0$$

Reálné řeš.: $e^{\lambda x}, xe^{\lambda x}, \dots, x^{k-1}e^{\lambda x}$

Komplexní řeš.: $e^{\alpha x}\cos\beta x, xe^{\alpha x}\cos\beta x, \dots, x^{k-1}e^{\alpha x}\cos\beta x$

$e^{\alpha x}\sin\beta x, xe^{\alpha x}\sin\beta x, \dots, x^{k-1}e^{\alpha x}\sin\beta x$

Ob. řeš. je součtem příslušných partikulárních řeš.: $y=C_1e^{\lambda x}+C_2xe^{\lambda x}+\dots+C_a x^{k-1}e^{\lambda x}+C_{a+1}e^{\alpha x}\cos\beta x+\dots$

$\dots+C_{a+k-1}x^{k-1}\cos\beta+C_{a+k}e^{\alpha x}\sin\beta x+\dots+C_{a+2k-1}x^{k-1}e^{\alpha x}\sin\beta x$

II, nehomogenní

a, metoda variace konstant

$$C_1'(x)y_1(x)+\dots C_n'(x)y_n(x)=0$$

$$C_1'(x)y_1'(x)+\dots+C_n'(x)y_n'(x)=0$$

$$\vdots$$

$$C_1'(x)y^{(n-1)}(x)+\dots+C_n'(x)y_n^{(n-1)}(x)=f(x)$$

15.3.2011

b, metoda neurčitých koeficientů

i) $f(x)=Q_m(x)e^{\alpha x}$ $\quad y_p=x^k\tilde{Q}_m(x)e^{\alpha x}$, kde

k je násobnost čísla α jako kořene char. polynomu

ii) $f(x)=e^{\alpha x}(P_m(x)\cos\beta x+Q_n(x)\sin\beta x)$

$y_p=x^k e^{\alpha x}(\tilde{P}_r(x)\cos\beta x+\tilde{Q}_r(x)\sin\beta x)$, kde k

je násobnost čísla $\alpha+\beta i$ jako kořene char. polynomu

a $r=\max(m;n)$

Platí-li $y^{(n)}+a_1y^{(n-1)}+\dots+a_{n-1}y'+a_ny=f_1(x)+f_2(x)$, pak

$y=y_{p_1}+y_{p_2}$, kde y_{p_1} je řeš. $y^{(n)}+a_1y^{(n-1)}+\dots+a_ny=f_1(x)$ a y_{p_2} je řeš. $y^{(n)}+a_1y^{(n-1)}+\dots+a_ny=f_2(x)$

5) $y'' + 3y' + 2y = (20x + 29)e^{3x}$, $\alpha = 3$

$\lambda^2 + 3\lambda + 2 = 0$

$$\lambda_{1,2} = \frac{-3 \pm \sqrt{3^2 + 4\cdot 2}}{2} = \begin{cases} \frac{-3+1}{2} = -1 \\ \frac{-3-1}{2} = -2 \end{cases}$$

$y_H = C_1 e^{-2x} + C_2 e^{-x}$

$y_P = (Ax + B)e^{3x}$

$y_P' = Ae^{3x} + (Ax+B)\cdot e^{3x}\cdot 3$

$y_P'' = Ae^{3x}\cdot 3 + 3Ae^{3x} + 3(Ax+B)\cdot e^{3x}\cdot 3 = 6Ae^{3x} + 9(Ax+B)e^{3x}$

$6Ae^{3x} + 9(Ax+B)e^{3x} + 3\cdot[Ae^{3x} + 3(Ax+B)e^{3x}] + 2(Ax+B)e^{3x} =$

$= (20x + 29)e^{3x}$

$\underline{6Ae^{3x}} + 9Axe^{3x} + \underline{9Be^{3x}} + \underline{3Ae^{3x}} + 9Axe^{3x} + \underline{9Be^{3x}} + 2Axe^{3x} + \underline{2Be^{3x}} =$

$= 20xe^{3x} + 29e^{3x}$

$6A + 9B + 3A + 9B + 2B = 29$

$9A + 9A + 2A = 20$

$9A + 20B = 29$

$20A = 20$

$A = 1 \quad B = 1$

$$y = C_1e^{-2x} + C_2e^{-x} + (x+1)e^{3x}, \quad C_1, C_2 \in R$$

6) $y'' - 2y' + 5y = 8e^{2x}\sin x$, $\alpha = 2 \quad \beta = 1$

$\lambda^2 - 2\lambda + 5 = 0$

$$\lambda_{1,2} = \frac{2 \pm \sqrt{4 - 4\cdot 5}}{2} = \frac{2 \pm \sqrt{16i^2}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

$y_H = C_1 e^{x}\cos 2x + C_2 e^{x}\sin 2x$

$$y_p = (A\sin x + B\cos x)e^{2x}$$

$$y_p' = (A\cos x - B\sin x)e^{2x} + 2(A\sin x + B\cos x)e^{2x}$$

$$y_p'' = (-A\sin x - B\cos x)e^{2x} + 2(A\cos x - B\sin x)e^{2x} +$$

$$+ 2(A\cos x - B\sin x)e^{2x} + 4(A\sin x + B\cos x)e^{2x} =$$

$$= -A\sin x e^{2x} - 2B\sin x e^{2x} - 2B\sin x e^{2x} + 4A\sin x e^{2x} -$$
$$- B\cos x e^{2x} + 2A\cos x e^{2x} + 2A\cos x e^{2x} + 4B\cos x e^{2x}$$

$$(-A-2B-2B+4A)\sin x e^{2x} + (-B+2A+2A+4B)\cos x e^{2x} -$$

$$-2\left[(A+2B)\cos x e^{2x} + (-B+2A)\sin x e^{2x}\right] +$$

$$+ 5A\sin x e^{2x} + 5B\cos x e^{2x} = 5e^{2x}\sin x$$

$$-A-2B-2B+4A+2B-4A+5A = 5$$

$$\underline{-B+2A+2A+4B-2A-4B+5B = 0}$$

$$4A - 2B = 5$$

$$\underline{2A + 4B = 0} \Rightarrow A = -2B$$

$$4(-2B) - 2B = 5$$

$$-8B - 2B = 5$$

$$-10B = 5$$

$$B = -\frac{1}{2}$$

$$A = 1$$

$$\underline{y = C_1 e^x \cos 2x + C_2 e^x \sin 2x + \left(\sin x - \frac{1}{2}\cos x\right)e^{2x}, \quad C_1, C_2 \in \mathbb{R}}$$

7. $y^{(5)} - 3y^{(4)} + 2y^{(3)} = 8x - 12$. $\acute{U}$

$\lambda^5 - 3\lambda^4 + 2\lambda^3 = 0$

$\lambda^3(\lambda^2 - 3\lambda + 2) = 0$

$\lambda_{1,2,3} = 0 \qquad \lambda_{4,5} = \frac{3 \pm \sqrt{9-4\cdot 2}}{2} = \left\langle \begin{matrix} 2 \\ 1 \end{matrix} \right.$

$y_H = c_1 + c_2 x + c_3 x^2 + c_4 e^{2x} + c_5 e^x$

$y_p = x^3(Ax + B) = Ax^4 + Bx^3$

$y_p' = 4Ax^3 + 3Bx^2$

$y_p'' = 12Ax^2 + 6Bx$

$y_p''' = 24Ax + 6B$

$y_p^{(4)} = 24A$

$y_p^{(5)} = 0$

$0 - 3\cdot 24A + 2\cdot(24Ax + 6B) = 8x - 12$

$-72A + 12B = -12$

$\underline{48A = 8}$

$A = \frac{1}{6}$

$\frac{12}{6}$
72

$-72\cdot\frac{1}{6} + 12B = -12$

$-12 + 12B = -12$

$B = 0$

$\underline{\underline{y = c_1 + c_2 x + c_3 x^2 + c_4 e^{2x} + c_5 e^x + \frac{1}{6}x^4, \; c_1, c_2, c_3, c_4, c_5 \in \mathbb{R}}}$

8. $y'' + y = tg^2 x$. $\acute{U}$

$\lambda^2 + 1 = 0$

$\lambda_{1,2} = \pm i$

$y_H = c_1 \sin x + c_2 \cos x$

$c_1'(x)\sin x + c_2'(x)\cos x = 0 \quad /\cdot \sin x$

$\underline{c_1'(x)\cos x - c_2'(x)\sin x = tg^2 x} \quad /\cdot \cos x$

$c_1'(x)(\sin^2 x + \cos^2 x) + c_2'(x)(\cos x \sin x - \sin x \cos x) = \frac{\sin^2 x}{\cos x}$

$c_1'(x) = \frac{\sin^2 x}{\cos x}$

$\frac{\sin^3 x}{\cos x} + c_2'(x)\cos x = 0 \Rightarrow c_2'(x) = -\frac{\sin^3 x}{\cos^2 x}$

$$C_1(x) = \int \frac{\sin^2 x}{\cos x}\,dx = \int \frac{1-\cos^2 x}{\cos x}\,dx = \int \frac{1}{\cos x}\,dx - \int \cos x\,dx =$$

$$= \int \frac{1}{\cos x}\,dx - \sin x = \int \frac{\cos x}{\cos^2 x}\,dx - \sin x = \left| \begin{array}{l} t = \sin x \\ dt = \cos x\,dx \end{array} \right| =$$

$$= \int \frac{dt}{1-t^2} - \sin x = \int \frac{-\frac{1}{2}}{t-1}\,dt + \int \frac{\frac{1}{2}}{t+1}\,dt - \sin x =$$

$$= -\frac{1}{2}\ln|t-1| + \frac{1}{2}\ln|t+1| - \sin x + C_1 =$$

$$= -\frac{1}{2}\ln|\sin x - 1| + \frac{1}{2}\ln|\sin x + 1| - \sin x + C_1$$

$$\frac{1}{(1-t)(1+t)} = \frac{A}{1-t} + \frac{B}{1+t}$$

$$1 = A(1+t) + B(1-t)$$

$$A + B = 1$$
$$A - B = 0$$
$$A = B$$
$$2A = 1$$
$$A = \frac{1}{2} \qquad B = \frac{1}{2}$$

$$C_2(x) = -\int \frac{\sin^3 x}{\cos^2 x}\,dx = -\int \frac{(1-\cos^2 x)\sin x}{\cos^2 x}\,dx =$$

$$= \left| \begin{array}{l} t = \cos x \\ dt = -\sin x\,dx \end{array} \right| = \int \frac{1-t^2}{t^2}\,dt = \int \frac{1}{t^2}\,dt - \int dt =$$

$$= -\frac{1}{t} - t + C_2 = -\frac{1}{\cos x} - \cos x + C_2$$

$$\cancel{y = \sin x\left(\frac{1}{2}\ln\left|\frac{\sin x+1}{\sin x-1}\right| - \sin x + C_1\right)}$$

$$y = C_1 \sin x + C_2 \cos x + \left(\frac{1}{2}\ln\left|\frac{\sin x + 1}{\sin x - 1}\right| - \sin x\right)\sin x +$$

$$+ \left(-\frac{1}{\cos x} - \cos x\right)\cos x =$$

$$= C_1 \sin x + C_2 \cos x + \frac{1}{2}\sin x \ln\left|\frac{\sin x+1}{\sin x-1}\right| - \sin^2 x - 1 - \cos^2 x =$$

$$= \underline{\underline{C_1 \sin x + C_2 \cos x + \frac{1}{2}\sin x \ln\left|\frac{\sin x+1}{\sin x-1}\right| - 2, \quad C_1, C_2 \in \mathbb{R}}}$$

9) $y'' + 4y = \frac{1}{\sin 2x}$

$$\begin{pmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{pmatrix} \begin{pmatrix} C_1'(x) \\ C_2'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{\sin 2x} \end{pmatrix}$$

$\lambda^2 + 4 = 0$

$\lambda = \pm 2i$

$y_H = C_1 \overbrace{\cos 2x}^{y_1} + C_2 \overbrace{\sin 2x}^{y_2}$

$C_1'(x)\cos 2x + C_2'(x)\sin 2x = 0 \qquad /\cdot 2\sin 2x$

$-2C_1'(x)\sin 2x + 2C_2'(x)\cos 2x = \frac{1}{\sin 2x} \qquad /\cdot \cos 2x$

$C_1'(x)(2\sin 2x\cos 2x - 2\sin 2x\cos 2x) + C_2'(x)(2\sin^2 2x + 2\cos^2 2x) = \frac{\cos 2x}{\sin 2x}$

$2C_2'(x) = \frac{\cos 2x}{\sin 2x}$

$C_2'(x) = \frac{1}{2}\frac{\cos 2x}{\sin 2x}$

$C_1'(x)\cos 2x + \frac{1}{2}\frac{\cos 2x}{\sin 2x}\cdot\sin 2x = 0$

$C_1'(x) = -\frac{1}{2}$

$C_1(x) = \int -\frac{1}{2}\,dx = -\frac{1}{2}x + C_1$

$C_2(x) = \frac{1}{2}\int \frac{\cos 2x}{\sin 2x}\,dx = \left| \begin{matrix} t = \sin 2x \\ dt = 2\cos 2x\,dx \end{matrix} \right| = \frac{1}{4}\int \frac{dt}{t} =$

$= \frac{1}{4}\ln|\sin 2x| + C_2$

$y = C_1\cos 2x + C_2\sin 2x + (-\frac{1}{2}x)\cos 2x + (\frac{1}{4}\ln|\sin 2x|)\sin 2x,$

$C_1, C_2 \in \mathbb{R}$

10, $y'' + y' = e^{0x}(x^2 - x) + 6e^{2x}$; $\alpha = 0$

$\alpha = 2$

$\lambda^2 + \lambda = 0$

$\lambda(\lambda + 1) = 0$

$\lambda_1 = 0$, $\lambda_2 = -1$ $k = 0$

$k = 1$

$y_H = C_1 + C_2 e^{-x}$

$y_{P_1} = x(Ax^2 + Bx + C)e^{0x} = Ax^3 + Bx^2 + Cx$

$y_{P_1}' = 3Ax^2 + 2Bx + C$

$y_{P_1}'' = 6Ax + 2B$

$6Ax + 2B + 3Ax^2 + 2Bx + C = x^2 - x$

$3A = 1$

$6A + 2B = -1$

$2B + C = 0$

$A = \frac{1}{3}$

$2 + 2B = -1$

$2B = -3$

$B = -\frac{3}{2}$

$-3 + C = 0$

$C = 3$

$y_{P_1} = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 3x$

$y_{P_2} = De^{2x}$

$y_{P_2}' = 2De^{2x}$

$y_{P_2}'' = 4De^{2x}$

$4De^{2x} + 2De^{2x} = 6e^{2x}$

$6D = 6$

$D = 1$

$y_{P_2} = e^{2x}$

$$\underline{\underline{y = C_1 + C_2 e^{-x} + \frac{1}{3}x^3 - \frac{3}{2}x^2 + 3x + e^{2x}, \quad C_1, C_2 \in \mathbb{R}}}$$

11) $y'' - 2y' + y = \frac{e^x}{x}$

$\lambda^2 - 2\lambda + 1 = 0$

$(\lambda - 1)^2 = 0$

$\lambda_{1,2} = 1$

$y_H = C_1 e^x + C_2 x e^x$

$$\begin{aligned} C_1'(x) e^x + C_2'(x) x e^x &= 0 \\ C_1'(x) e^x + C_2'(x) e^x + C_2'(x) x e^x &= \frac{e^x}{x} \end{aligned}$$

$C_1'(x) = -C_2'(x) x$

$-C_2'(x) x e^x + C_2'(x) e^x + C_2'(x) x e^x = \frac{e^x}{x}$

$C_2'(x) e^x = \frac{e^x}{x}$

$C_2'(x) = \frac{1}{x}$

$C_1'(x) = -1$

$C_1(x) = \int -1\,dx = -x + C_1$

$C_2(x) = \int \frac{1}{x}\,dx = \ln|x| + C_2$

$\underline{\underline{y = C_1 e^x + C_2 x e^x + (-x) e^x + (\ln|x|) x e^x, \quad C_1, C_2 \in \mathbb{R}}}$

12) $y'' + 2y' + 2y = 3e^{-x}\cos x \qquad (-1+i)$

$\lambda^2 + 2\lambda + 2 = 0$

$\lambda_{1,2} = \frac{-2 \pm \sqrt{4 - 4\cdot 2}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$

$y_H = C_1 e^{-x}\cos x + C_2 e^{-x}\sin x$

$y_P = x e^{-x}(A\cos x + B\sin x) =$ ~~[illegible]~~

~~[illegible]~~

$y_P' = e^{-x}(A\cos x + B\sin x) + x e^{-x}(-1)(A\cos x + B\sin x) +$
$+ x e^{-x}(-A\sin x + B\cos x)$

$y_P'' = -e^{-x}(A\cos x + B\sin x) + e^{-x}(-A\sin x + B\cos x) +$
$+ (e^{-x} + x e^{-x}(-1))(-1)(A\cos x + B\sin x) - x e^{-x}(-A\sin x + B\cos x) +$
$+ (e^{-x} - x e^{-x})(-A\sin x + B\cos x) + x e^{-x}(-A\cos x - B\sin x)$

$-Ae^{-x}\cos x - Be^{-x}\sin x - Ae^{-x}\sin x + Be^{-x}\cos x - e^{-x}A\cos x - e^{-x}B\sin x +$
$+ Axe^{-x}\cos x + Bxe^{-x}\sin x + Axe^{-x}\sin x - Bxe^{-x}\cos x - Ae^{-x}\sin x +$
$+ Be^{-x}\cos x + Axe^{-x}\sin x - Bxe^{-x}\cos x - Axe^{-x}\cos x - Bxe^{-x}\sin x +$
$+ 2Ae^{-x}\cos x + 2Be^{-x}\sin x - 2Axe^{-x}\cos x - 2xe^{-x}B\sin x -$
$- 2Axe^{-x}\sin x + 2Bxe^{-x}\cos x + 2xe^{-x}A\cos x + 2xe^{-x}B\sin x =$
$= 3e^{-x}\cos x$

$$-A + B - A + B + 2A = 3$$

$$-B - A - B - A + 2B = 0$$

$$A - B - B - A - 2A + 2B + 2A = 0$$

$$B + A + A - B - 2B - 2A + 2B = 0$$

$$2B = 3$$

$$-2A = 0$$

$$A = 0$$

$$B = \frac{3}{2}$$

$$y = C_1 e^{-x} \cos x + C_2 e^{-x} \sin x + x e^{-x}\left(\frac{3}{2} \sin x\right), \; C_1, C_2 \in \mathbb{R}$$

Eulerova dif. rovnice

$$a_0 x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \dots + a_{n-1} x y' + a_n y = f(x)$$

$$a_0, a_1, \dots, a_n \in \mathbb{R}, \quad a_0 \neq 0$$

subst.: $t = \ln|x|$

$$y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \dot{y} \frac{1}{x}$$

$$y'' = \frac{dy'}{dx} = \frac{dy'}{dt} \cdot \frac{dt}{dx} = \frac{d\dot{y}}{dt} \cdot \frac{dt}{dx} \cdot \frac{1}{x} + \dot{y} \frac{d}{dx}\left(\frac{1}{x}\right) =$$

$$= \ddot{y} \frac{1}{x^2} + \dot{y}\left(-\frac{1}{x^2}\right) = \ddot{y} \frac{1}{x^2} - \frac{1}{x^2} \dot{y}$$

$$y^{(k)} = \left[\frac{d}{dt}\left(\frac{d}{dt} - 1\right) \dots \left(\frac{d}{dt} - k + 1\right)\right] y \cdot x^{-k}$$

$k = 1$: $y' = \frac{dy}{dt} \cdot x^{-1} = \dot{y} \frac{1}{x}$

$k = 2$: $y'' \left[\frac{d}{dt}\left(\frac{d}{dt} - 1\right)\right] y \cdot x^{-2} = \left(\frac{d^2 y}{dt} - \frac{dy}{dt}\right) x^{-2} =$

$$= (\ddot{y} - \dot{y}) \frac{1}{x^2}$$

$k = 3$: $y''' = \left[\frac{d}{dt}\left(\frac{d}{dt} - 1\right)\left(\frac{d}{dt} - 2\right)\right] y x^{-3} =$

$$= \left[\left(\frac{d^2}{dt} - \frac{d}{dt}\right)\left(\frac{d}{dt} - 2\right)\right] y x^{-3} =$$

$$= \left(\frac{d^3 y}{dt} - 2\frac{d^2 y}{dt} - \frac{d^2 y}{dt} + 2\frac{dy}{dt}\right) x^{-3} =$$

$$= \left(\dddot{y} - 3\ddot{y} + 2\dot{y}\right) \frac{1}{x^3}$$

1) $x^2 y'' - 6x y' + 6y = x^2$

$t = \ln|x| \qquad x = e^t \Rightarrow x^2 = e^{2t}$

$y' = \dot{y} \frac{1}{x} \qquad y'' = (\ddot{y} - \dot{y}) \frac{1}{x^2}$

$$\ddot{y} - \dot{y} - 6\dot{y} + 6y = e^{2t}$$

$$\ddot{y} - 7\dot{y} + 6y = e^{2t}$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$\lambda_{1,2} = \frac{7 \pm \sqrt{49 - 24}^{\,5}}{2} = \Big\langle {6 \atop 1}$$

$$y_H(t) = C_1 e^{6t} + C_2 e^t$$

$$\left.\begin{aligned} z(t) &= A e^{2t} \\ z'(t) &= A2e^{2t} \\ z''(t) &= A4e^{2t} \end{aligned}\right\} \Rightarrow \qquad \begin{aligned} 4Ae^{2t} - 14Ae^{2t} + 6Ae^{2t} &= e^{2t} \\ 4A - 14A + 6A &= 1 \\ -4A &= 1 \\ A &= -\frac{1}{4} \end{aligned}$$

$$y(t) = C_1 e^{6t} + C_2 e^t - \frac{1}{4} e^{2t} = C_1 x^6 + C_2 x - \frac{1}{4} x^2$$

2) $y'' - \frac{y'}{x} + \frac{y}{x^2} = \frac{2}{x}$

$x^2 y'' - x y' + y = 2x$

$t = \ln|x| \Rightarrow e^t = |x|$

$y' = \dot{y}\,\frac{1}{x} \qquad y'' = (\ddot{y} - \dot{y})\,\frac{1}{x^2}$

$\ddot{y} - \dot{y} - \dot{y} + y = 2e^t \operatorname{sgn} x$

$\ddot{y} - 2\dot{y} + y = 2e^t \operatorname{sgn} x$

$\lambda^2 - 2\lambda + 1 = 0$

$\lambda_{1/2} = 1$

$y_H(t) = C_1 e^t + C_2 t e^t$

$z(t) = A e^t t^2$

$z'(t) = A e^t t^2 + A e^t 2t$

$z''(t) = A e^t \cdot t^2 + A e^t \cdot 2t + A e^t \cdot 2t + A e^t \cdot 2$

$A t^2 e^t + 2Ate^t + 2Ate^t + 2Ae^t - 2At^2 e^t - 4tAe^t +$

$+ Ae^t t^2 = 2e^t \operatorname{sgn} x$

$t^2 e^t (A - 2A + A) = 0$

$te^t (2A + 2A - 4A) = 0$

$2Ae^t = 2e^t \operatorname{sgn} x$

$A = \operatorname{sgn} x$

$y(t) = C_1 e^t + C_2 t e^t + \operatorname{sgn} x \cdot e^t t^2 =$

$= C_1 |x| + C_2 |x| \ln|x| + \operatorname{sgn} x\, |x| \cdot \ln^2|x|$