CVIČENÍ 1

G 125

Polynom: $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

$a_0, \dots, a_n \in \mathbb{R}$ jsou koeficienty

$a_n \neq 0$, n je stupeň polynomu a odpovídá mu počet kořenů polynomu

Interpolace: postup, pomocí kterého hledáme polynom procházející danými body

1, najděte polynom procházející danými body

a, $[-2; -1]$; $[1, 5]$ metoda neurčitých koeficientů

$f(x) = ax + b$

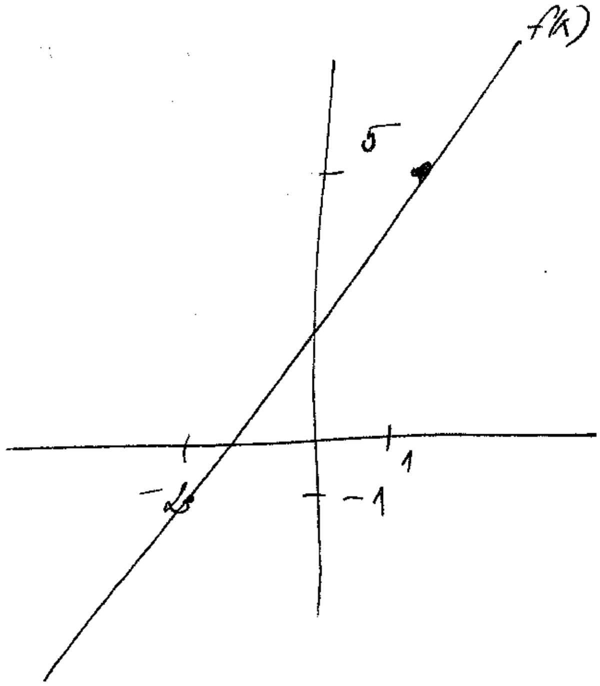
$-1 = a \cdot (-2) + b$
$5 = a \cdot 1 + b$

$b = -1 + 2a$
$b = 5 - a$

$-1 + 2a = 5 - a$
$3a = 6$
$\underline{\underline{a = 2}}$

$b = -1 + 2a$
$b = -1 + 4$
$\underline{\underline{b = 3}}$

$f(x) = 2x + 3$



b) $[-2; 4]; [0, -2], [1, -2]$

$f(x) = ax^2 + bx + c$

$4 = a(-2)^2 + b(-2) + c$

$-2 = a(0)^2 + b\cdot 0 + c$

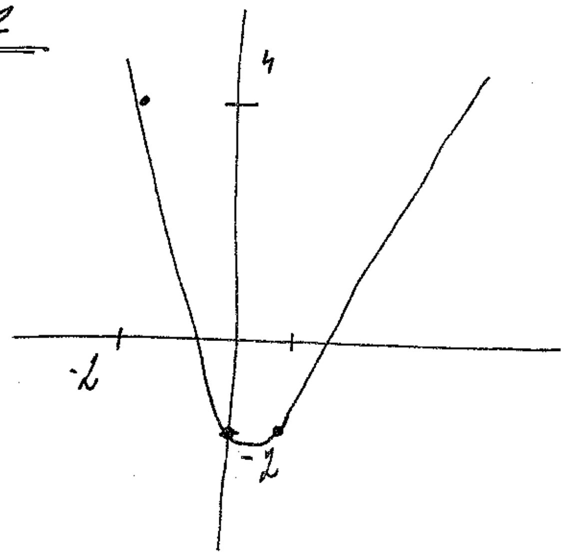
$-2 = a\cdot 1^2 + b\cdot 1 + c$

$$\left(\begin{array}{ccc|c} 4 & -2 & 1 & 4 \\ 0 & 0 & 1 & -2 \\ 1 & 1 & 1 & -2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -6 & -3 & 12 \\ 0 & 0 & 1 & -2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & 1 & \frac{1}{2} & -2 \\ 0 & 0 & 1 & -2 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & +1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

$\underline{a = 1}$ $\underline{b = -1}$ $\underline{c = -2}$

$f(x) = x^2 - x - 2$



c) $[-1, 1]; [0, 1], [1, 5], [2, 31]$

$f(x) = ax^3 + bx^2 + cx + d$

$1 = -1a + 1b + (-1)c + d$

$1 = 0a + 0b + 0c + d$

$5 = 1a + 1b + 1c + d$

$31 = 8a + 4b + 2c + d$

$$\left(\begin{array}{cccc|c} -1 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 5 \\ 8 & 4 & 2 & 1 & 31 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & -1 \\ 0 & 2 & 0 & 2 & 6 \\ 0 & 12 & -6 & 9 & 39 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & -1 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 4 & -2 & 3 & 13 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & -1 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & -2 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$a = 3 \quad b = 2 \quad c = -1 \quad d = 1$

$f(x) = 3x^3 + 2x^2 \cancel{x} - x + 1$

<u>Lagrangeův interpolační polynom</u>

pomocí polynomy $l_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

$$l_i(x) = \frac{\prod_{j \neq i} (x - x_j)}{\prod_{j \neq i} (x_i - x_j)}$$

Lagrangeův interpolační polynom bude mít tvar

$$f(x) = y_0 l_0(x) + y_1 l_1(x) + \dots + y_n l_n(x)$$

d) $[-1, 1], [0, 1]; [1; 5]; [2; 31]$

$i = 0 :$ ~~[illegible]~~ $x_0 = -1 \quad y_0 = 1 \quad l_0(\cancel{-0}) = 0 \quad \cancel{l_0(-1)} = 1$

$l_0(1) = 0$

$l_0(2) = 0$

$$f(x) = 1 \cdot \frac{(x-0)(x-1)(x-2)}{(-1-0)(-1-1)(-1-2)} + 1 \cdot \frac{(x-(-1))(x-1)(x-2)}{(0-(-1))(0-1)(0-2)} +$$

$$+ 5 \cdot \frac{(x+1)(x-0)(x-2)}{(1-(-1))(1-0)(1-2)} + 31 \cdot \frac{(x+1)(x-0)(x-1)}{(2+1)(2-0)(2-1)} =$$

$$= \frac{x(x^2 - 2x - x + 2)}{(-1)\cdot(-2)(-3)} + \frac{(x^2-1)(x-2)}{1\cdot(-1)\cdot(-2)} + \frac{5x(x^2 + x - 2x - 2)}{2\cdot 1\cdot(-1)} +$$

$$+ \frac{31x(x^2-1)}{3\cdot 2\cdot 1} = -\frac{1}{6}\left(x^3 - 3x^2 + 2x\right) + \frac{1}{2}\left(x^3 - 2x^2 - x + 2\right) +$$

$$- \frac{5}{2}\left(x^3 - x^2 - 2x \right) + \frac{31}{6}\left(x^3 - x \right) =$$

$$= x^3\left(-\frac{1}{6} + \frac{1}{2} - \frac{5}{2} + \frac{31}{6} \right) + x^2\left(-\frac{1}{6}\cdot(-3) - 1 + \frac{5}{2} \right) +$$

$$+ x\left(-\frac{1}{3} - \frac{1}{2} + 5 - \frac{31}{6} \right) + 1 =$$

$$= \underline{\underline{3x^3 + 2x^2 - 1x + 1}}$$

e) $[-1; 2]; [0, 1], [1, 0], [2; 5]$

$$L(x) = 2 \cdot \frac{(x-0)(x-1)(x-2)}{(-1-0)(-1-1)(-1-2)} + 1 \cdot \frac{(x+1)(x-1)(x-2)}{(0+1)(0-1)(0-2)} +$$

$$+ 0 \cdot \frac{\cdots}{\cdots} + 5 \cdot \frac{(x+1)(x-0)(x-1)}{(2+1)(2-0)(2-1)} =$$

$$= \frac{2x(x^2 - 3x + 2)}{-1\cdot(-2)(-3)} + \frac{(x^2-1)(x-2)}{1(-1)(-2)} + \frac{5x(x^2-1)}{3\cdot 2\cdot 1} =$$

$$= -\frac{1}{3}\left(x^3 - 3x^2 + 2x \right) + \frac{1}{2}\left(x^3 - 2x^2 - x + 2 \right) + \frac{5}{6}\left(x^3 - x \right) =$$

$$= x^3\left(-\frac{1}{3} + \frac{1}{2} + \frac{5}{6} \right) + x^2\left(1 - 1 \right) + x\left(-\frac{2}{3} - \frac{1}{2} - \frac{5}{6} \right) + 1 =$$

$$= 1x^3 + 0x^2 - 2x + 1 = x^3 - 2x + 1$$

Hermiteův polynom

jsou zadány nejen body, kterými prochází, ale i jeho derivace v některých bodech

2) Najděte Hermiteův interpolační polynom splňující podm.

a) $f(-1)=0$, $f(1)=2$, $f'(\frac{1}{2})=3$

$f(x)=ax^2+bx+c$ $\qquad$ $f'(x)=2ax+b$

$0=a-b+c$ $\qquad$ $3=2a(\frac{1}{2})+b$

$2=a+b+c$ $\qquad$ $3=a+b$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 3 \end{array}\right) \sim \cancel{\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & +2 & +0 & +2 \\ 0 & 2 & 0 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 1 & 0 & 3 \end{array}\right)}$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & -1 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & +1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$f(x)=2x^2+x-1$

b) $f(1)=-1$ $\quad$ $f(0)=-2$ $\quad$ $f'(2)=12$ $\quad$ $f'(1)=3$

$f(x)=ax^3+bx^2+cx+d$ $\qquad$ $f'(x)=3ax^2+2bx+c$

$-1=a+b+c+d$ $\qquad$ $12=3a\cdot 4+2b\cdot 2+c$

$-2=d$ $\qquad$ $3=3a\cdot 1+2b\cdot 1+c$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -1 \\ 12 & 4 & 1 & 0 & 12 \\ 3 & 2 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right) \sim \cancel{\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -1 \\ 0 & -8 & -11 & -12 & 0 \\ 0 & -4 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right)} \sim \cancel{\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -1 \\ 0 & 8 & 11 & 0 & 0 \\ 0 & 4 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right)}$$

$\cancel{8b+11c=0}$

$\cancel{4b+3c=0}$

$\cancel{8b+11c=4b+3c}$

$\cancel{4b=-8c}$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -1 \\ 3 & 2 & 1 & 0 & 3 \\ 12 & 4 & 1 & 0 & 12 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -1 \\ 0 & -1 & -2 & -3 & 6 \\ 0 & -8 & -11 & -12 & 24 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & 3 & -6 \\ 0 & 0 & 5 & 12 & -24 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right) \quad \begin{array}{l} b = 0 \\ c = 0 \\ d = -2 \end{array}$$

$$a + 0 + 0 - 2 = -1 \Rightarrow a = 1$$

$$f(x) = x^3 - 2$$

Rozklad na parciální zlomky

$P(x), Q(x)$ ... polynomy

$$R(x) = \frac{P(x)}{Q(x)} \quad \text{... racionální lomená f-ce}$$

- α ... reálný kořen s násobností k

$$\frac{A_1}{(x-\alpha)^k} + \frac{A_2}{(x-\alpha)^{k-1}} + \dots + \frac{A_k}{x-\alpha}$$

- $\beta \pm \gamma i$ ... dvojice komplexně sdružených kořenů s násobností k

$$\frac{B_1 x + C_1}{(x^2+px+q)^k} + \frac{B_2 x + C_2}{(x^2+px+q)^{k-1}} + \dots + \frac{B_k x + C_k}{(x^2+px+q)}$$

3, Rozložte na parciální zlomky

a) $\dfrac{3x^2-5x+8}{x^3-2x^2+x-2}$

$$=\frac{3x^2-5x+8}{x^2(x-2)+x-2}=\frac{3x^2-5x+8}{(x^2+1)(x-2)}$$

$$\frac{3x^2-5x+8}{(x^2+1)(x-2)}=\frac{Ax+B}{x^2+1}+\frac{C}{x-2}$$

$$3x^2-5x+8=(Ax+B)(x-2)+C(x^2+1)$$

$$3x^2-5x+8=Ax^2+Bx-2Ax-2B+Cx^2+C$$

$$3x^2-5x+8=(A+C)x^2+(B-2A)x-2B+C$$

$$A+C=3$$
$$B-2A=-5$$
$$-2B+C=8$$

$$A=1 \qquad B=-3 \qquad C=2$$

$$\frac{3x^2-5x+8}{(x^2+1)(x-2)}=\frac{x-3}{x^2+1}+\frac{2}{x-2}$$

a) $\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)}$

$$\frac{1}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

$$1 = A(x^2-x+1) + (Bx+C)(x+1)$$

$$1 = Ax^2 - Ax + A + Bx^2 + Bx + Cx + C$$

$$1 = (A+B)x^2 + (-A+B+C)x + A+C$$

$A+B=0$

$-A+B+C=0$

$A+C=1$

$A = \frac{1}{3}$

$B = -\frac{1}{3}$

$C = \frac{2}{3}$

б) $\frac{1}{x^3(x+1)}$

$$\frac{1}{x^3(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+1}$$

$$1 = Ax^2(x+1) + Bx(x+1) + C(x+1) + Dx^3$$

$$1 = (A+D)x^3 + (B+A)x^2 + (B+C)x + C$$

$A+D=0$

$A+B=0$

$B+C=0$

$C=1$

$A=1 \quad B=-1 \quad C=1 \quad D=-1$

$$d) \quad \frac{x^2 - x + 10}{(x^2 - 3x + 10)^2}$$

$$\frac{x^2 - x + 10}{(x^2 - 3x + 10)^2} = \frac{Ax + B}{(x^2 - 3x + 10)} + \frac{Cx + D}{(x^2 - 3x + 10)^2}$$

$$x^2 - x + 10 = (Ax + B)(x^2 - 3x + 10)^2 + Cx + D$$

$$x^2 - x + 10 = Ax^3 + 3Ax^2 + 10Ax + Bx^2 - 3Bx + 10B + Cx + D$$

$$x^2 - x + 10 = Ax^3 + (3A + B)x^2 + (10A - 3B + C)x + 10B + D$$

$$A = 0$$
$$3A + B = 1$$
$$10A \cancel{+} - 3B + C = -1$$
$$10B + D = 10$$

$$A = 0 \qquad B = 1 \qquad C = 2 \qquad D = 0$$

$$e) \quad \frac{x^2 - 2}{x^4 - 2x^3 + 2x^2}$$

$$\frac{x^2 - 2}{x^2(x^2 - 2x + 2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 - 2x + 2}$$

$$x^2 - 2 = Ax^3 - 2Ax^2 + 2Ax + Bx^2 - 2Bx + 2B + Cx^3 + Dx^2$$

$$x^2 - 2 = (A + C)x^3 + (-2A + B + D)x^2 + (2A - 2B)x + 2B$$

$$A + C = 0 \qquad A = -1$$
$$-2A + B + D = 1 \qquad B = -1$$
$$2A - 2B = 0 \qquad C = 1$$
$$2B = -2 \qquad D = 0$$