CVIČENÍ 2

Supremum: $A \subseteq \mathbb{R}$, $b \in \mathbb{R}$ je supremum, pokud $\forall x \in A: x \leq b$

Infimum: $B \subseteq \mathbb{R}$, $c \in \mathbb{R}$, $c = \inf B \iff \forall x \in B: x \geq c$

Okolí bodu: $$O(x_0) = \begin{cases} (x_0 - \delta, x_0 + \delta) & x_0 \in \mathbb{R} \\ (a, \infty) & x_0 = \infty \\ (-\infty, b) & x_0 = -\infty \end{cases}$$

Ryzí prstencové okolí bodu x_0: $O(x_0) \setminus \{x_0\}$

Limita: Buď $x_0, L \in \mathbb{R}$. Fce $f(x)$ má v bodě x_0 limitu L, $\lim\limits_{x \to x_0} f(x) = L$, pokud pro každé okolí $O(L)$ existuje okolí $O(x_0)$ tak, že

$$\forall x \in O(x_0) \setminus \{x_0\} \text{ je } f(x) \in O(L).$$

Vlastnosti limit: Je-li $\lim\limits_{x \to x_0} f(x) = L$, $\lim\limits_{x \to x_0} g(x) = M$, pak

1) $\lim\limits_{x \to x_0} [f(x) \pm g(x)] = L \pm M$

2) $\lim\limits_{x \to x_0} f(x) \cdot g(x) = L \cdot M$

3) $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, pokud $M \neq 0$

4) $\lim\limits_{x \to x_0} |f(x)| = |\lim\limits_{x \to x_0} f(x)| = |L|$

Spojitost: Fce $f(x)$ je spojitá v bodě $x_0 \in \mathbb{R}$, jestliže v tomto bodě existuje vlastní limita L, v bodě x_0 existuje funkční hodnota $f(x_0)$ a $f(x_0) = L$.

Vlastnosti spojitých f-cí: Je-li $f(x)$ a $g(x)$ spojitá, pak

$$(f \pm g)(x), \quad f(x)\cdot g(x), \quad \frac{f(x)}{g(x)} \text{ jsou také spojité}$$

Je-li $\lim_{x\to x_0} g(x) = M$, $f(y)$ je spojitá v bodě $y_0 = M$, pak $\lim_{x\to x_0} f(g(x)) = f(\lim_{x\to x_0} g(x)) = f(M)$

Je-li $g(x)$ spojitá v x_0, $f(y)$ spojitá v $y_0 = g(x_0)$, pak $(f\circ g)(x) = f(g(x))$ je spojitá v x_0.

- $\lim_{x\to\pm\infty} \left(1+\frac{1}{x}\right)^x = e$
- $\lim_{x\to 0} \frac{\sin x}{x} = 1$
- $\lim_{x\to\infty} \frac{a^x - 1}{x} = \ln a$
- $\lim_{x\to\infty} \frac{\sin x}{x} = 0$
- $\lim_{x\to\infty} \frac{e^x - 1}{x} = 1$

Limity posloupností a řad

1, $\lim_{n\to\infty} \frac{2^n - 3^n}{3^n} = \lim_{n\to\infty} \left[\left(\frac{2}{3}\right)^n - 1^n\right] = \lim_{n\to\infty} \underbrace{\left(\frac{2}{3}\right)^n}_{\to 0} - 1 = \underline{\underline{-1}}$

2, $\lim_{n\to\infty} \left(\sqrt{(n+a)(n+b)} - n\right) = \lim_{n\to\infty} \frac{(n+a)(n+b) - n^2}{\sqrt{(n+a)(n+b)} + n} =$

$= \lim_{n\to\infty} \frac{n^2 + (a+b)n + ab - n^2}{\sqrt{(n+a)(n+b)} + n} = \lim_{n\to\infty} \frac{(a+b)n + ab}{\sqrt{(n+a)(n+b)} + n} =$

$= \lim_{n\to\infty} \frac{\cancel{n}\left(a+b+\frac{ab}{n}\right)}{\cancel{n}\left(\sqrt{1+\frac{a+b}{n}+\frac{ab}{n^2}} + 1\right)} = \underline{\underline{\frac{a+b}{2}}}$

(where $\frac{ab}{n} \to 0$, $\frac{a+b}{n} \to 0$, $\frac{ab}{n^2} \to 0$)

3, $\lim_{n\to\infty} \frac{3n^2+1}{3n+n^2} = \lim_{n\to\infty} \frac{\cancel{n^2}\left(3+\frac{1}{n^2}\right)}{\cancel{n^2}\left(\frac{3}{n}+1\right)} = \underline{\underline{3}}$

(where $\frac{1}{n^2} \to 0$, $\frac{3}{n} \to 0$)

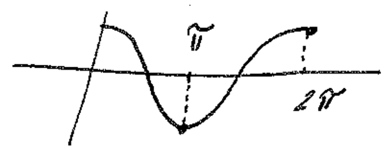
4, $\lim\limits_{n\to\infty} \left(\sqrt{n+\sqrt{n}} - \sqrt{2n+1}\right) = \lim\limits_{n\to\infty} \dfrac{n+\sqrt{n}-(2n+1)}{\sqrt{n+\sqrt{n}}+\sqrt{2n+1}} =$

$= \lim\limits_{n\to\infty} \dfrac{-n-1+\sqrt{n}}{\sqrt{n+\sqrt{n}}+\sqrt{2n+1}} = \lim\limits_{n\to\infty} \dfrac{\sqrt{n}\left(-\sqrt{n}-\overset{\nearrow 0}{\dfrac{1}{\sqrt{n}}}+1\right)}{\sqrt{n}\left(\sqrt{1+\underset{\downarrow 0}{\dfrac{1}{\sqrt{n}}}}\right)+\sqrt{n}\left(\sqrt{2+\underset{\downarrow 0}{\dfrac{1}{n}}}\right)} =$

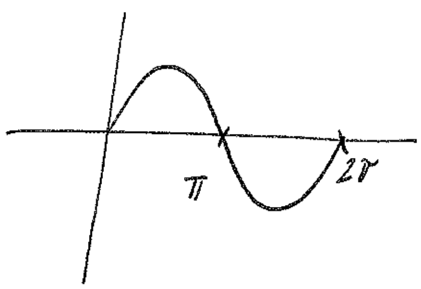
$= \lim\limits_{n\to\infty} \dfrac{-\sqrt{n}+1}{1+\sqrt{2}} = \underline{\underline{-\infty}}$

5, $\lim\limits_{n\to\infty} \cos\dfrac{1}{n} = \cos \lim\limits_{n\to\infty} \dfrac{1}{n} = \cos 0 = \underline{\underline{1}}$

6, $\lim\limits_{n\to\infty} (1+\cos n\pi)$ diverguje



7, $\lim\limits_{n\to\infty} (1+\sin n\pi) = 1+\lim\limits_{n\to\infty} \sin n\pi = 1+0 = \underline{\underline{1}}$



8, $\lim\limits_{n\to\infty} \dfrac{\sin n}{n} = 0$

9, $\lim\limits_{n\to\infty} \sqrt[n]{5n} = \lim\limits_{n\to\infty} \sqrt[n]{5} \cdot \underbrace{\lim\limits_{n\to\infty} \sqrt[n]{n}}_{=1} = \underline{\underline{1}}$

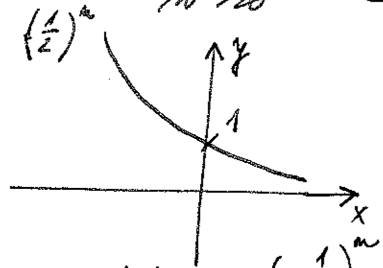
10, $\lim\limits_{n\to\infty} \sqrt[2n]{n} = \lim\limits_{n\to\infty} \sqrt[n]{n^{\frac{1}{2}}} = \left(\lim\limits_{n\to\infty} \sqrt[n]{n}\right)^{\frac{1}{2}} = 1^{\frac{1}{2}} = \underline{\underline{1}}$

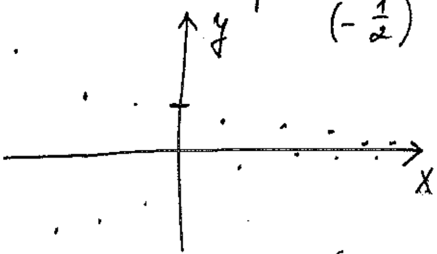
11, $\lim\limits_{n\to\infty} (n^2-5n-1) = \lim\limits_{n\to\infty} n^2\left(1-\dfrac{5}{n}-\dfrac{1}{n^2}\right) = \infty$

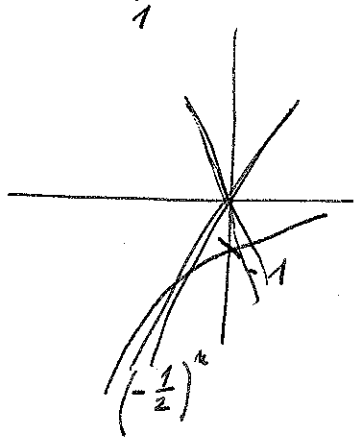
12, $\lim\limits_{n\to\infty} \dfrac{-8n^2+6n+7}{2n+5} = \lim\limits_{n\to\infty} \dfrac{n\left(-8n+6+\dfrac{7}{n}\right)}{n\left(2+\dfrac{5}{n}\right)} = \underline{\underline{-\infty}}$

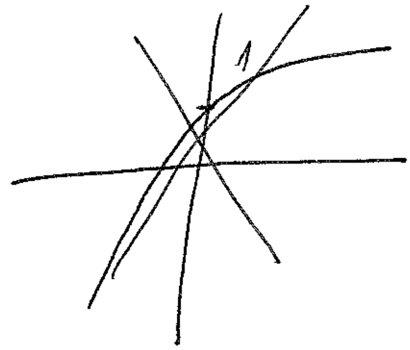
13) $$\lim_{n\to\infty} \frac{2^n + (-2)^n}{2\cdot 4^n} = \lim_{n\to\infty} \frac{2^n(1+(-1)^n)}{2^n(2\cdot 2^n)} = \lim_{n\to\infty} \frac{1+(-1)^n}{2\cdot 2^n} =$$

$$= \frac{1}{2}\lim_{n\to\infty} \underbrace{\left(\left(\frac{1}{2}\right)^n + \left(-\frac{1}{2}\right)^n\right)}_{1} = \frac{1}{2}$$









14) $$\lim_{n\to\infty} \frac{(n+2)! - 3n!}{(n+2)!+1} = \lim_{n\to\infty} \frac{(n+2)(n+1)n! - 3n!}{(n+2)(n+1)n! + 1} =$$

$$= \lim_{n\to\infty} \frac{n![(n+2)(n+1)-3]}{n![(n+2)(n+1)+\frac{1}{n!}]} = \lim_{n\to\infty} \frac{n^2+3n+2-3}{n^2+3n+2+\frac{1}{n!}} =$$

$$= \lim_{n\to\infty} \frac{n^2\left(1+\frac{3}{n}-\frac{1}{n^2}\right)}{n^2\left(1+\frac{3}{n}+\frac{2}{n^2}+\frac{1}{n^2 n!}\right)} = \frac{1}{1} = \underline{\underline{1}}$$

Limity funkcí:

15) $$\lim_{x\to 1}\left(\frac{1}{-x+1} - \frac{3}{1-x^3}\right) = \lim_{x\to 1} \frac{1+x+x^2-3}{1-x^3} =$$

$$= \lim_{x\to 1}\left(\frac{x^2+x-2}{1-x^3}\right) = \lim_{x\to 1} \frac{(x+2)(x-1)}{(1-x)(1+x+x^2)} = \lim_{x\to 1} \frac{-x-2}{1+x+x^2} = \underline{\underline{-1}}$$

16) $$\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x^2-5x+6} = \lim_{x\to 3} \frac{\sqrt{x+1}-2}{(x-3)(x-2)} = \lim_{x\to 3} \frac{x+1-4}{(x-3)(x-2)(\sqrt{x+1}+2)} =$$

$$= \lim_{x\to 3} \frac{x-3}{(x-3)(x-2)(\sqrt{x+1}+2)} = \lim_{x\to 3} \frac{1}{(x-2)(\sqrt{x+1}+2)} = \underline{\underline{\frac{1}{4}}}$$

17) $$\lim_{x\to 0} \frac{tg x - \sin x}{\sin^3 x} = \lim_{x\to 0} \left(\frac{\sin x}{\cos x} - \frac{\sin x \cdot \cos x}{\cos x} \right) \cdot \frac{1}{\sin^3 x} =$$

$$= \lim_{x\to 0} \frac{\sin x (1-\cos x)}{\cos x \cdot \sin^3 x} = \lim_{x\to 0} \frac{(1-\cos x)}{\cos x \cdot \sin^2 x} =$$

$$= \lim_{x\to 0} \frac{(1-\cos x)}{\cos x \cdot (1-\cos^2 x)} = \lim_{x\to 0} \frac{1-\cos x}{\cos x (1+\cos x)(1-\cos x)} =$$

$$= \lim_{x\to 0} \frac{1}{\cos x (1+\cos x)} = \frac{1}{1\cdot(1+1)} = \underline{\underline{\frac{1}{2}}}$$

18) $$\lim_{x\to 0} \frac{tg x}{x} = \lim_{x\to 0} \frac{\sin x}{\cos x} \cdot \frac{1}{x} = \lim_{x\to 0} \underbrace{\frac{\sin x}{x}}_{=1} \cdot \frac{1}{\cos x} =$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{1}{\cos x} = 1 \cdot \frac{1}{\cos 0} = \underline{\underline{1}}$$