CVIČENÍ 3

Body nespojitosti:

• odstranitelná nespojitost : $\exists$ vlastní limita $\lim\limits_{x\to x_0} f(x)$, ale $f(x_0) \neq \lim\limits_{x\to x_0} f(x)$ nebo $f(x_0)$ není definována

lze ji odstranit předefinováním hodnoty v x_0

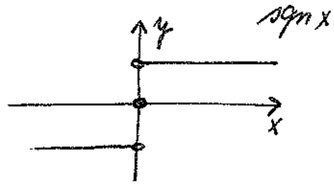
Př.: $f(x) = \dfrac{x^2-9}{x+3}$

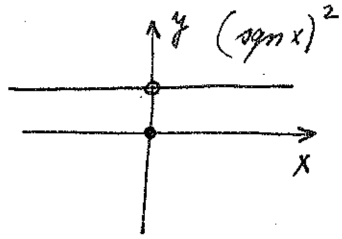
$x_0 = -3$

$$\lim_{x\to -3} \frac{x^2-9}{x+3} = \lim_{x\to -3} \frac{(x+3)(x-3)}{x+3} = \lim_{x\to -3} (x-3) = -3-3 = -6$$

$f(x_0) = f(-3) = -6$

Př.: $f(x) = (\operatorname{sgn} x)^2$



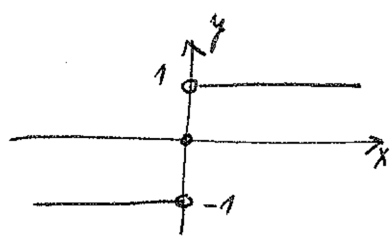


$x_0 = 0$

$f(x_0) = 1$

• nespojitost 1. druhu (skok): existují vlastní limity $\lim\limits_{x\to x_0^+} f(x)$, $\lim\limits_{x\to x_0^-} f(x)$ a $\lim\limits_{x\to x_0^-} f(x) \neq \lim\limits_{x\to x_0^+} f(x)$

Př.: $f(x) = \operatorname{sgn} x$

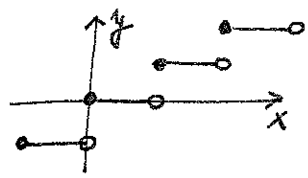


$x_0 = 0$

$\lim\limits_{x\to 0^+} \operatorname{sgn} x = 1$

$\lim\limits_{x\to 0^-} \operatorname{sgn} x = -1$

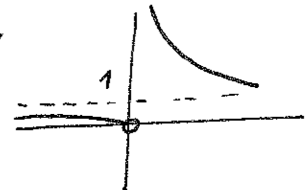
Př.: $f(x) = [x]$



• nespojitost 2. druhu : alespoň jedna jednostranná limita je buď nevlastní nebo neexistuje

Př.: $f(x) = \dfrac{1}{x}$ $x_0 = 0$

Př.: $f(x) = e^{\frac{1}{x}}$



1) najděte body nespojitosti a určete jejich druh

a) $f(x) = \dfrac{\sin x}{|x|}$

$x_0 = 0$

$$\left.\begin{aligned} \lim_{x\to 0^+} \frac{\sin x}{x} &= 1 \\ \lim_{x\to 0^-} \frac{\sin x}{-x} &= -1 \end{aligned}\right\} \Rightarrow \text{nespojitost 1. druhu}$$

b) $f(x) = \dfrac{x^3-1}{x^2-1} = \dfrac{(x-1)(x^2+x+1)}{(x-1)(x+1)}$

$x_0 = 1$

$x_0' = -1$

$$\lim_{x\to 1} \frac{x^2+x+1}{x+1} = \frac{3}{2} \Rightarrow \text{bod odstranitelné nespojitosti}$$

$$\left.\begin{aligned} \lim_{x\to -1^-} \frac{x^2+x+1}{x+1} &= \frac{1}{0^-} = -\infty \\ \lim_{x\to -1^+} \frac{x^2+x+1}{x+1} &= \frac{1}{0^+} = \infty \end{aligned}\right\} \text{bod nespojitosti 2. druhu}$$

Limity

2) $$\lim_{x\to 0} \frac{1-\cos 2x + \operatorname{tg}^2 x}{x\cdot \sin x} = \lim_{x\to 0} \frac{1-(\cos^2 x - \sin^2 x) + \frac{\sin^2 x}{\cos^2 x}}{x\cdot\sin x} =$$

$$= \lim_{x\to 0} \frac{1-\cos^2 x + \sin^2 x + \frac{\sin^2 x}{\cos^2 x}}{x\cdot \sin x} = \lim_{x\to 0} \frac{\sin^2 x + \sin^2 x + \frac{\sin^2 x}{\cos^2 x}}{x\cdot\sin x} =$$

$$= \lim_{x\to 0} \frac{2\sin^2 x + \frac{\sin^2 x}{\cos^2 x}}{x\cdot \sin x} = \lim_{x\to 0}\left(2\cdot \frac{\sin^2 x}{x\cdot\sin x} + \frac{\sin^2 x}{\cos^2 x}\cdot\frac{1}{x\cdot\sin x}\right) =$$

$$= \lim_{x\to 0}\left(2\cdot\frac{\sin x}{x} + \frac{\sin x}{x}\cdot\frac{1}{\cos^2 x}\right) = 2+1 = \underline{\underline{3}}$$

3) $\lim\limits_{x\to 0} \dfrac{\sin kx}{x} = \lim\limits_{x\to 0} \dfrac{k\cdot \sin kx}{kx} = k\cdot \lim\limits_{x\to 0} \dfrac{\sin kx}{kx} = \underline{\underline{k}}$

4) $\lim\limits_{x\to\infty} 2^{\frac{3x}{x+2}} = 2^{\lim\limits_{x\to\infty} \frac{3x}{x+2}} = 2^{\lim\limits_{x\to\infty} \frac{x(3)}{x(1+\frac{2}{x})}} = 2^{3} = \underline{\underline{8}}$

5) $\lim\limits_{x\to 0} \left(\dfrac{\sin 2x}{x}\right)^{1+x} = \lim\limits_{x\to 0} e^{(1+x)\ln\left(\frac{\sin 2x}{x}\right)} =$

$= e^{\lim\limits_{x\to 0} (1+x)\ln\left(\frac{2\sin 2x}{2x}\right)} = e^{\lim\limits_{x\to 0} (1+0)\ln 2\cdot 1} = e^{\ln 2} = \underline{\underline{2}}$

6) $\lim\limits_{x\to 0} \dfrac{\sin 4x + \sin 7x}{\sin 3x} = \lim\limits_{x\to 0} \left(\dfrac{\sin 4x}{\sin 3x} + \dfrac{\sin 7x}{\sin 3x}\right) =$

$= \lim\limits_{x\to 0} \left(\dfrac{\sin 4x}{4x}\cdot\dfrac{3x}{\sin 3x}\cdot\dfrac{4x}{3x} + \dfrac{\sin 7x}{7x}\cdot\dfrac{3x}{\sin 3x}\cdot\dfrac{7x}{3x}\right) =$

$= \lim\limits_{x\to 0} \dfrac{\sin 4x}{4x}\cdot \lim\limits_{x\to 0} \dfrac{1}{\frac{\sin 3x}{3x}}\cdot\dfrac{4}{3} + \lim\limits_{x\to 0} \dfrac{\sin 7x}{7x}\cdot$

$\cdot \lim\limits_{x\to 0} \dfrac{1}{\frac{\sin 3x}{3x}}\cdot\dfrac{7}{3} = 1\cdot 1\cdot \dfrac{4}{3} + 1\cdot 1\cdot\dfrac{7}{3} = \underline{\underline{\dfrac{11}{3}}}$

7) $\lim\limits_{x\to\infty} \dfrac{\sqrt{1+x} - \sqrt{1+x^2}}{\sqrt{1+x} - 1} = \lim\limits_{x\to\infty} \dfrac{\sqrt{x}\left(\overset{\to 1}{\sqrt{\frac{1}{x}+1}} - \overset{\to\infty}{\sqrt{\frac{1}{x}+x}}\right)}{\sqrt{x}\left(\underset{\downarrow 1}{\sqrt{\frac{1}{x}+1}} - \underset{\downarrow 0}{\frac{1}{\sqrt{x}}}\right)} =$

$= -\underline{\underline{\infty}}$

8) $$\lim_{x\to 3}\frac{\sqrt{x+13}-2\sqrt{x+1}}{x^2-9}=\lim_{x\to 3}\frac{x+13-4(x+1)}{(x+3)(x-3)(\sqrt{x+13}+2\sqrt{x+1})}=$$

$$=\lim_{x\to 3}\frac{-3x+9}{(x+3)(x-3)(\sqrt{x+13}+2\sqrt{x+1})}=\lim_{x\to 3}\frac{-3(x-3)}{(x+3)(x-3)(\sqrt{x+13}+2\sqrt{x+1})}=$$

$$=\lim_{x\to 3}\frac{-3}{(x+3)(\sqrt{x+13}+2\sqrt{x+1})}=\frac{-3}{6(\sqrt{16}+2\sqrt{4})}=-\underline{\underline{\frac{1}{16}}}$$

9) $$\lim_{x\to\infty}\frac{\sqrt{x^2+1}+\sqrt{x}}{\sqrt[4]{x^3+x}-x}=\lim_{x\to\infty}\frac{x\left(\sqrt{1+\frac{1}{x^2}}+\sqrt{\frac{1}{x}}\right)}{x\left(\sqrt{\frac{1}{x}+\frac{1}{x^3}}-1\right)}=\frac{1}{-1}=\underline{\underline{-1}}$$

10) $$\lim_{x\to 0}\frac{\sin 4x}{\sqrt{1+x}-1}=\lim_{x\to 0}\frac{\sin 4x(\sqrt{1+x}+1)}{1+x-1}=\lim_{x\to 0}\frac{\sin 4x}{x}(\sqrt{1+x}+1)=$$

$$=\lim_{x\to 0}4\frac{\sin 4x}{4x}(\sqrt{1+x}+1)=4\cdot(\sqrt{1+0}+1)=\underline{\underline{8}}$$

11) $$\lim_{x\to 0}\frac{\ln(1+ax)}{x}=\lim_{x\to 0}\ln(1+ax)^{\frac{1}{x}}=\ln\lim_{x\to 0}(1+ax)^{\frac{1}{x}}=$$

$$=\left|\begin{matrix}z=\frac{1}{x}\\ x\to 0\Leftrightarrow z\to\infty\end{matrix}\right|=\ln\lim_{z\to\infty}\left(1+\frac{a}{z}\right)^z=\left|\begin{matrix}y=\frac{z}{a}\\ z=ya\end{matrix}\right|=$$

$$=\ln\lim_{y\to\infty}\left[\left(1+\frac{1}{y}\right)^y\right]^a=\ln e^a=\underline{\underline{a}}$$

12) $$\lim_{x\to\infty}x\left(\operatorname{arctg}\frac{x+1}{x+2}-\frac{\pi}{4}\right)=*$$

$$\lim_{x\to\infty}\operatorname{arctg}\frac{x+1}{x+2}-\operatorname{arctg}1=\lim_{x\to\infty}\operatorname{arctg}\frac{x+1-x-2}{x+2}\cdot\frac{x+2}{x+2+x+1}=$$

$$\operatorname{arctg}x-\operatorname{arctg}y=\operatorname{arctg}\frac{x-y}{1+xy}$$

$$=\lim_{x\to\infty}\operatorname{arctg}\frac{-1}{2x+3}$$

substituce: $\operatorname{tg} z=\frac{1}{2x+3}$

$$\frac{1}{tg\, z} = 2x+3$$

$$x = \frac{1}{2}\left(\frac{1}{tg\, z} - 3\right)$$

$$\frac{1}{2x+3} \xrightarrow{x\to\infty} 0 \Rightarrow tg\, z \longrightarrow 0 \Rightarrow z \to 0$$

$$* = \lim_{z\to 0} \frac{1}{2}\left(\frac{1}{tg\, z} - 3\right)\cdot(-z) = \frac{1}{2}\lim_{z\to 0}\left(\frac{-z}{tg\, z} + 3z\right) =$$

$$= \frac{1}{2}\lim_{z\to 0}\left(\frac{-z}{\frac{\sin z}{\cos z}} + 3z\right) = \frac{1}{2}\lim_{z\to 0}\left(-\frac{z}{\sin z}\cdot \cos z - 3z\right) = -\frac{1}{2}$$

13) $$\lim_{x\to\infty}\left(\frac{x-1}{x+1}\right)^x = \lim_{x\to\infty}\frac{\left(1-\frac{1}{x}\right)^x}{\left(1+\frac{1}{x}\right)^x} = \lim_{x\to\infty}\frac{\left[\left(1-\frac{1}{x}\right)^{-x}\right]^{-1}}{\left(1+\frac{1}{x}\right)^x} =$$

$$= \frac{e^{-1}}{e} = e^{-2}$$

14) $$\lim_{x\to\infty}\frac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}} = \lim_{x\to\infty}\frac{\sqrt{x}}{\sqrt{x}\left(\sqrt{1+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x^3}}}}\right)} = \lim_{x\to\infty}\frac{1}{\sqrt{1+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x^3}}}}} =$$

$$= 1$$

15) $$\lim_{x\to\frac{\pi}{2}}\frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos x} = \lim_{x\to\frac{\pi}{2}}\frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}} = \lim_{x\to\frac{\pi}{2}}\frac{1}{\cos\frac{x}{2}+\sin\frac{x}{2}} =$$

$$= \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)^{-1} = \frac{1}{\frac{2\sqrt{2}}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

16) $$\lim_{x\to\frac{\pi}{6}}\frac{\sin\left(x-\frac{\pi}{6}\right)}{\frac{\sqrt{3}}{2} - \cos x} = \lim_{x\to\frac{\pi}{6}}\frac{\sin\left(x-\frac{\pi}{6}\right)}{\cos\frac{\pi}{6} - \cos x} = \left|\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}\right|$$

$$= \lim_{x\to\frac{\pi}{6}}\frac{\sin\left(x-\frac{\pi}{6}\right) + \sin 0}{-2\sin\frac{\frac{\pi}{6}-x}{2}\sin\frac{\frac{\pi}{6}-x}{2}} = \left|\sin\alpha + \sin\beta\right| = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\Big| =$$

$$= \lim_{x\to\frac{\pi}{6}}\frac{2\sin\frac{x-\frac{\pi}{6}}{2}\cos\frac{x-\frac{\pi}{6}}{2}}{-2\sin\frac{x+\frac{\pi}{6}}{2}\sin\frac{\frac{\pi}{6}-x}{2}} = \lim_{x\to\frac{\pi}{6}}\frac{\cos\frac{x-\frac{\pi}{6}}{2}}{\sin\frac{x+\frac{\pi}{6}}{2}} = \frac{1}{\frac{1}{2}} = 2$$

17) $\lim\limits_{x \to \frac{\pi}{4}} \dfrac{\cos 2x - \sin 2x + 1}{\cos x - \sin x} = \lim\limits_{x \to \frac{\pi}{4}} \dfrac{\cos^2 x - \sin^2 x - 2\sin x \cos x + \sin^2 x + \cos^2 x}{\cos x - \sin x} = \lim\limits_{x \to \frac{\pi}{4}} \dfrac{2\cos^2 x - 2\sin x \cos x}{\cos x - \sin x} = \lim\limits_{x \to \frac{\pi}{4}} \dfrac{2\cos x(\cos x - \sin x)}{\cos x - \sin x} =$

$= \lim\limits_{x \to \frac{\pi}{4}} 2\cos x = 2 \cdot \dfrac{\sqrt{2}}{2} = \underline{\underline{\sqrt{2}}}$

18) $\lim\limits_{x \to 0} \dfrac{e^{2x} - 1}{x} = \left| \begin{matrix} z = 2x \\ x = \frac{z}{2} \end{matrix} \right| = \lim\limits_{z \to 0} \dfrac{e^{z} - 1}{\frac{z}{2}} = 2 \lim\limits_{z \to 0} \dfrac{e^{z} - 1}{z} =$

$= 2 \cdot 1 = \underline{\underline{2}}$

19) $\lim\limits_{x \to 0} \sqrt[x]{\cos x + x + 2} = e^{\lim\limits_{x \to 0} \ln \sqrt[x]{\cos x + x + 2}} =$

$= e^{\lim\limits_{x \to 0} \frac{1}{x} \ln(\cos x + x + 2)} = \begin{cases} x \to 0^- & = -\infty \\ x \to 0^+ & = \infty \end{cases}$