CVIČENÍ 4

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Pravidla pro derivování:

$$c' = 0$$
$$(x^n)' = n x^{n-1}$$
$$(e^x)' = e^x$$
$$(a^x)' = a^x \ln a$$
$$(\ln x)' = \frac{1}{x}$$
$$(\log_a x)' = \frac{1}{x \ln a}$$
$$(\sin x)' = \cos x$$
$$(\cos x)' = -\sin x$$
$$(\mathrm{tg}\, x) = \frac{1}{\cos^2 x}$$
$$(\mathrm{cotg}\, x)' = \frac{-1}{\sin^2 x}$$
$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$
$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$
$$(\mathrm{arctg}\, x)' = \frac{1}{1+x^2}$$
$$(\mathrm{arccotg}\, x)' = -\frac{1}{1+x^2}$$

Když má fce v bodě x_0 vlastní derivaci, pak je v tomto bodě spojitá.

$f'(x_0) = \mathrm{tg}\, \alpha$ ... derivace určuje směrnici tečny ke grafu fce f v bodě x_0

$$[f(x) \pm g(x)]' = f'(x) \pm g'(x)$$
$$[c \cdot f(x)]' = c \cdot f'(x)$$

$$(f \cdot g)' = f' \cdot g + f \cdot g'$$
$$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$$
$$[f(g(x))]' = f'(g(x)) \cdot g'(x)$$

1) $(6x)' = 6$

2) $(x^2)' = 2x$

3) $(\sqrt{x})' = (x^{\frac{1}{2}})' = \frac{1}{2}x^{-\frac{1}{2}}$

4) $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$

5) $\left(\sqrt[4]{x^7}\right)' = \left(x^{\frac{7}{4}}\right)' = \frac{7}{4}x^{\frac{3}{4}}$

6) $(x^3+2x-\sin x+2)' = 3x^2+2-\cos x$

7) $(xe^x)' = e^x + xe^x = (1+x)e^x$

8) $\left(\frac{3x-2}{x^2+1}\right)' = \frac{3(x^2+1)-(3x-2)(2x)}{(x^2+1)^2} = \frac{-3x^2+4x+3}{(x^2+1)^2}$

9) $[(3x^2-2x+10)^{10}]' = 10\cdot(3x^2-2x+10)^9\cdot(3\cdot 2x-2)$

10) $\left(\sqrt{4-x^2}\right)' = \frac{1}{2}(4-x^2)^{-\frac{1}{2}}(-2x) = \frac{-x}{\sqrt{4-x^2}}$

11) $(\ln \sin x)' = \frac{1}{\sin x}\cdot\cos x = \mathrm{cotg}\, x$

12) $\left(\sqrt{\sin 3x}\right)' = \frac{1}{2}(\sin 3x)^{-\frac{1}{2}}\cdot\cos 3x\cdot 3 = \frac{3}{2}\frac{\cos 3x}{\sqrt{\sin 3x}}$

13) $(x^x)' = \left(e^{\ln x^x}\right)' = \left(e^{x\ln x}\right)' = e^{x\ln x}\cdot\ln e\left(\ln x + x\frac{1}{x}\right) =$

$= e^{x\ln x}(\ln x+1) = x^x(\ln x+1)$

14) $(x^{x^2})' = \left(e^{x^2\ln x}\right)' = e^{x^2\ln x}\ln e\left(2x\ln x + x^2\frac{1}{x}\right) = x^{x^2}(2x\ln x + x) =$

$= x^{x^2+1}(2\ln x + 1)$

15) $(x^{\sin x})' = \left(e^{\sin x\ln x}\right)' = e^{\sin x\ln x}\cdot\ln e\left(\cos x\ln x + \sin x\cdot\frac{1}{x}\right) =$

$= x^{\sin x}\left(\cos x\ln x + \frac{1}{x}\sin x\right)$

16) $\left[(\sin x)^{\ln x}\right]' = \left[e^{\ln(\sin x)^{\ln x}}\right]' = \left(e^{\ln x\ln\sin x}\right)' = e^{\ln x\ln\sin x}\ln e\cdot\left(\frac{1}{x}\ln\sin x + \right.$

$\left. + \ln x\frac{1}{\sin x}\cos x\right) = (\sin x)^{\ln x}\left(\frac{1}{x}\ln\sin x + \frac{\ln x\cdot\cos x}{\sin x}\right)$

17) $\left(\sqrt{\frac{1-e^x}{1+e^x}}\right)' = \frac{1}{2}\frac{1}{\sqrt{\frac{1-e^x}{1+e^x}}}\cdot\frac{-e^x(1+e^x)-(1-e^x)e^x}{(1+e^x)^2} = \frac{-e^x-e^{2x}-e^x+e^{2x}}{2\sqrt{\frac{1-e^x}{1+e^x}}\,(1+e^x)^2} =$

$= \frac{-2e^x}{2\sqrt{\frac{1-e^x}{1+e^x}}\,(1+e^x)^2} = \frac{1}{2}\sqrt{\frac{1+e^x}{1-e^x}}\cdot\frac{(-2e^x)}{(1+e^x)^2} = -\frac{e^x\sqrt{1+e^x}}{(1+e^x)^2\sqrt{1-e^x}}$

18) $\left[(x^2+1)^{\operatorname{arctg} x}\right]' = \left(e^{\operatorname{arctg} x \ln(x^2+1)}\right)' = (x^2+1)^{\operatorname{arctg} x}\left(\frac{1}{1+x^2}\ln(x^2+1)+\right.$

$\left.+\operatorname{arctg} x\,\frac{1}{x^2+1}2x\right) = (x^2+1)^{\operatorname{arctg} x}\frac{1}{x^2+1}\left(\ln(x^2+1)+2x\operatorname{arctg} x\right) =$

$= (x^2+1)^{\operatorname{arctg} x - 1}\left(\ln(x^2+1)+2x\operatorname{arctg} x\right)$

19) $\left(\ln\frac{e^x}{x^2+1}\right)' = \frac{x^2+1}{e^x}\,\frac{e^x(x^2+1)-e^x2x}{(x^2+1)^2} = \frac{x^2-2x+1}{x^2+1} = \frac{(x-1)^2}{x^2+1}$

20) $\left(\ln\frac{\sqrt{x^2+1}}{x+1}\right)' = \frac{x+1}{\sqrt{x^2+1}}\cdot\frac{\frac{1}{2}\frac{2x}{\sqrt{x^2+1}}(x+1)-\sqrt{x^2+1}}{(x+1)^2} = \frac{\frac{2x(x+1)-2(x^2+1)}{2\sqrt{x^2+1}}}{\sqrt{x^2+1}\,(x+1)} =$

$= \frac{2x^2+2x-2x^2-2}{\sqrt{x^2+1}\,(x+1)2\sqrt{x^2+1}} = \frac{x-2}{(x+1)(x^2+1)}$

21) $\left(\ln\sqrt{\frac{1-\sin x}{1+\sin x}}\right)' = \sqrt{\frac{1+\sin x}{1-\sin x}}\cdot\frac{1}{2\sqrt{\frac{1-\sin x}{1+\sin x}}}\cdot\frac{-\cos x(1+\sin x)-(1-\sin x)\cos x}{(1+\sin x)^2} =$

$= \sqrt{\frac{1+\sin x}{1-\sin x}}\,\frac{\sqrt{1+\sin x}}{2\sqrt{1-\sin x}}\cdot\frac{-\cos x-\sin x\cos x-\cos x+\sin x\cos x}{(1+\sin x)^2} =$

$= \frac{1+\sin x}{2(1-\sin x)}\cdot\frac{-2\cos x}{(1+\sin x)^2} = \frac{-\cos x}{1-\sin^2 x} = -\frac{1}{\cos x}$

22) $\left(\ln\frac{x+2-2\sqrt{x+1}}{x}\right)' = \frac{x}{x-2-2\sqrt{x+1}}\cdot\frac{\left(1-\frac{1}{\sqrt{x+1}}\right)x-(x+2-2\sqrt{x+1})}{x^2} =$

$= \frac{x-\frac{x}{\sqrt{x+1}}-x-2+2\sqrt{x+1}}{(x-2-2\sqrt{x+1})x} = \frac{x\sqrt{x+1}-x-x\sqrt{x+1}-2\sqrt{x+1}+2(x+1)}{\sqrt{x+1}}\cdot$

$\cdot\frac{1}{x(x+2-2\sqrt{x+1})} = \frac{x+2-2\sqrt{x+1}}{x\sqrt{x+1}\,(x+2-2\sqrt{x+1})} = \frac{1}{x\sqrt{x+1}}$

23)
$$\left(\frac{\arccos x}{x} + \frac{1}{2}\ln\frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}}\right)' = \frac{\frac{-1}{\sqrt{1-x^2}}x - \arccos x\cdot 1}{x^2} +$$

$$+\frac{1}{2}\,\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\cdot\frac{-\frac{1}{\cancel{2}}\frac{1\cdot\cancel{2x}}{\sqrt{1-x^2}}\left(1+\sqrt{1-x^2}\right) + \left(1-\sqrt{1-x^2}\right)\frac{1}{2}\frac{1\cdot 2x}{\cancel{2}\sqrt{1-x^2}}}{\left(1+\sqrt{1-x^2}\right)^2} =$$

$$= \frac{-x}{x^2\sqrt{1-x^2}} - \frac{\arccos x}{x^2} + \frac{1}{2\left(1-\sqrt{1-x^2}\right)}\cdot\frac{\frac{x\left(1+\sqrt{1-x^2}\right)+x\left(1-\sqrt{1-x^2}\right)}{\sqrt{1-x^2}}}{\left(1+\sqrt{1-x^2}\right)} =$$

$$= \frac{-1}{x\sqrt{1-x^2}} - \frac{\arccos x}{x^2} + \frac{x+x\sqrt{1-x^2}+x-x\sqrt{1-x^2}}{2\left(1-\sqrt{1-x^2}\right)\sqrt{1-x^2}} =$$

$$= -\frac{1}{x\sqrt{1-x^2}} - \frac{\arccos x}{2} + \frac{\cancel{2}x}{\cancel{2}\sqrt{1-x^2}\,x^{\cancel{2}}} = -\frac{1}{x\sqrt{1-x^2}} - \frac{\arccos x}{2} + \frac{1}{x\sqrt{1-x^2}} =$$

$$= -\frac{\arccos x}{2}$$

24)
$$\left[(x+2)\sqrt{1+e^x} - \ln\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right]' = 1\cdot\sqrt{1+e^x} + (x-2)\frac{1}{2\sqrt{1+e^x}}e^x -$$

$$-\frac{\sqrt{1+e^x}+1}{\sqrt{1+e^x}-1}\cdot\frac{\frac{1}{2\sqrt{1+e^x}}e^x\left(\sqrt{1+e^x}+1\right) - \left(\sqrt{1+e^x}-1\right)\frac{1}{2\sqrt{1+e^x}}e^x}{\left(\sqrt{1+e^x}+1\right)^2} =$$

$$= \sqrt{1+e^x} + \frac{e^x(x-2)}{2\sqrt{1+e^x}} - \frac{\frac{e^x\left(1+\sqrt{1+e^x}\right) - e^x\left(\sqrt{1+e^x}-1\right)}{2\sqrt{1+e^x}}}{\left(\sqrt{1+e^x}+1\right)\left(\sqrt{1+e^x}-1\right)} = \sqrt{1+e^x} + \frac{e^x(x-2)}{2\sqrt{1+e^x}} -$$

$$-\frac{e^x+e^x\sqrt{1+e^x}-e^x\sqrt{1+e^x}+e^x}{2\sqrt{1+e^x}\left(1+e^x-1\right)} = \sqrt{1+e^x} + \frac{e^x(x-2)}{2\sqrt{1+e^x}} - \frac{\cancel{2e^x}}{\cancel{2e^x}\sqrt{1+e^x}} =$$

$$= \sqrt{1+e^x} + \frac{xe^x-2e^x-2}{2\sqrt{1+e^x}} * = \frac{2\left(1+e^x\right)+xe^x-2e^x-2}{2\sqrt{1+e^x}} = \frac{2+2e^x+xe^x-2e^x-2}{2\sqrt{1+e^x}} =$$

$$= \frac{xe^x}{2\sqrt{1+e^x}}$$

25) $$\left(\arcsin\frac{\sqrt{1-x}}{\sqrt{1+x}}\right)' = \frac{1}{\sqrt{1-\frac{1-x}{1+x}}}\cdot\frac{1}{2}\,\frac{1}{\sqrt{\frac{1-x}{1+x}}}\cdot\frac{-1(1+x)-(1-x)}{(1+x)^2} =$$

$$= \frac{1}{\sqrt{1-\frac{1-x}{1+x}}}\;\frac{\sqrt{1+x}}{2\sqrt{1-x}}\cdot\frac{-1-x-1+x}{(1+x)^2} = \frac{1}{\sqrt{\frac{1+x-1+x}{1+x}}}\;\frac{\sqrt{1+x}}{2\sqrt{1-x}}\;\frac{-2}{(1+x)^2} =$$

$$= \frac{\sqrt{1+x}}{\sqrt{2x}}\;\frac{\sqrt{1+x}}{2\sqrt{1-x}}\;\frac{-2}{(1+x)^2} = \frac{-1}{(1+x)\sqrt{2x(1-x)}}$$

26) $$(x^2\sin\sqrt{x})' = 2x\sin\sqrt{x} + x^2\cos\sqrt{x}\cdot\frac{1}{2}\,\frac{1}{\sqrt{x}} = 2x\sin\sqrt{x} + \frac{x^2\cos\sqrt{x}}{2\sqrt{x}}$$

27) $$(x^2\sin\sqrt{x})'' = \left(2x\sin\sqrt{x} + \frac{x^2\cos\sqrt{x}}{2\sqrt{x}}\right)' = 2\sin\sqrt{x} + 2x\cos\sqrt{x}\cdot\frac{1}{2}\,\frac{1}{\sqrt{x}} +$$

$$+ \frac{\left(2x\cos\sqrt{x} + x^2(-\sin\sqrt{x})\frac{1}{2\sqrt{x}}\right)2\sqrt{x} - x^2\cos\sqrt{x}\;2\,\frac{1}{2\sqrt{x}}}{4x} =$$

$$= 2\sin\sqrt{x} + \sqrt{x}\cos\sqrt{x} + \frac{1}{4x}\left(4x\sqrt{x}\cos\sqrt{x} - x^2\sin\sqrt{x} - x\sqrt{x}\cos\sqrt{x}\right) =$$

$$= 2\sin\sqrt{x} + \sqrt{x}\cos\sqrt{x} + \sqrt{x}\cos\sqrt{x} - \frac{x}{4}\sin\sqrt{x} - \frac{\sqrt{x}}{4}\cos\sqrt{x} =$$

$$= \sin\sqrt{x}\left(2-\frac{x}{2}\right) + \sqrt{x}\cos\sqrt{x}\left(2-\frac{1}{4}\right) = \sin\sqrt{x}\left(2-\frac{x}{4}\right) + \frac{7}{4}\sqrt{x}\cos\sqrt{x}$$

28) $$(\mathrm{tg}^2 x)' = \left(\frac{\sin^2 x}{\cos^2 x}\right)' = \frac{2\sin x\cos x\cdot\cos^2 x + \sin^2 x\,2\cos x\sin x}{\cos^4 x} =$$

$$= \frac{2\sin x\cos x(\cos^2 x + \sin^2 x)}{\cos^4 x} = \frac{2\sin x}{\cos^3 x}$$

29) $$(\mathrm{tg}^2 x)'' = \left(\frac{2\sin x}{\cos^3 x}\right)' = \frac{2\cos x\cdot\cos^3 x + 2\sin x\cdot 3\cos^2 x\cdot\sin x}{\cos^6 x} =$$

$$= \frac{2\cos^4 x + 6\sin^2 x\cos^2 x}{\cos^6 x} = \frac{\cos^2 x(2\cos^2 x + 6\sin^2 x)}{\cos^6 x} = \frac{2\cos^2 x + 6\sin^2 x}{\cos^4 x}$$