CVIČENÍ 6

1) Určete průběh f-ce $f(x) = \dfrac{x}{x^2+1}$

$D(f) = \mathbb{R} \Rightarrow$ žádné body nespojitosti

f je spojitá

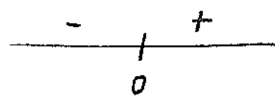
$f(-x) = \dfrac{-x}{x^2+1} \neq f(x) \Rightarrow$ není sudá

$-f(-x) = \dfrac{x}{x^2+1} = f(x) \Rightarrow$ f-ce je lichá

$f(x) = 0 \iff x = 0$

$x^2+1 > 0 \quad \forall x \in \mathbb{R} \Rightarrow f(x) > 0 \quad \forall x > 0$

$f(x) < 0 \quad \forall x < 0$



$$f'(x) = \frac{x^2+1-x(2x)}{(x^2+1)^2} = \frac{x^2-2x^2+1}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}$$

$$\frac{-x^2+1}{(x^2+1)^2} = 0$$

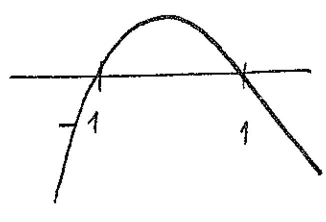
$$1-x^2 = 0$$

$$(1-x)(1+x) = 0$$

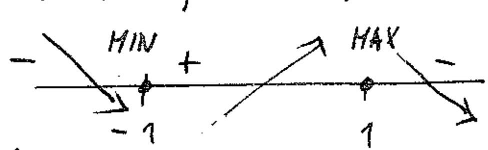
$x = \pm 1$ .... stacionární body

$f'(x) > 0 \iff \dfrac{1-x^2}{(x^2+1)^2} > 0 \iff 1-x^2 > 0$

$\iff x \in (-1; 1) \Rightarrow$ rostoucí



$f'(x) < 0 \iff 1-x^2 < 0 \iff x \in (-\infty; -1) \cup (1; \infty) \Rightarrow$ klesající



$$f''(x) = \left(\frac{-x^2+1}{(x^2+1)^2}\right)' = \frac{-2x(x^2+1)^2 - (1-x^2)\cdot 2(x^2+1)\cdot 2x}{(x^2+1)^4} =$$

$$= \frac{-2x\left[(x^2+1)^2 + 2(x^2+1)(1-x^2)\right]}{(x^2+1)^4} = \frac{-2x(x^2+1)\left[x^2+1+2(1-x^2)\right]}{(x^2+1)^4} =$$

$$= \frac{-2x(x^2+1+2-2x^2)}{(x^2+1)^3} = \frac{-2x(-x^2+3)}{(x^2+1)^3} = \frac{+2x(x^2-3)}{(x^2+1)^3}$$

$f''(x) = 0 \iff 2x(x^2-3) = 0$

$x = 0 \lor x = \pm\sqrt{3}$

$- \cap \quad -\sqrt{3} \quad + \cup \quad 0 \quad - \cap \quad \sqrt{3} \quad + \cup$

$f''(x) > 0 \iff x \in (-\sqrt{3}; 0) \cup (\sqrt{3}, \infty)$ → konvexní

$f''(x) < 0 \iff x \in (-\infty; -\sqrt{3}) \cup (0, \sqrt{3})$ → konkávní

asymptoty bez směrnice nejsou, prot. nejsou žádné body nespojitosti

$$a_1 = \lim_{x\to\infty} \frac{\frac{x}{x^2+1}}{x} = \lim_{x\to\infty} \frac{1}{x^2+1} = 0$$

$$a_2 = \lim_{x\to-\infty} \frac{\frac{x}{x^2+1}}{x} = \lim_{x\to-\infty} \frac{1}{x^2+1} = 0$$

$$b_1 = \lim_{x\to\infty} \left(\frac{x}{x^2+1} - 0x\right) = \lim_{x\to\infty} \frac{1}{x+\frac{1}{x}} = 0$$

$$b_2 = \lim_{x\to-\infty} \frac{1}{x+\frac{1}{x}} = 0$$

asymptota se směrnicí: $y = 0$

významné body: $[0;0]$, stacionární − $\underset{\text{minimum}}{[-1; -\frac{1}{2}]}$, $\underset{\text{maximum}}{[1; \frac{1}{2}]}$

inflexní − $[-\sqrt{3}; -\frac{\sqrt{3}}{4}]$, $[0, 0]$, $[\sqrt{3}, \frac{\sqrt{3}}{4}]$

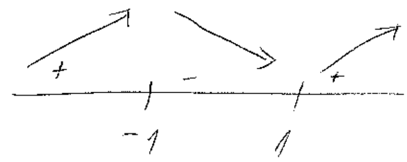
$$\lim_{x\to\infty} \frac{x}{x^2+1} = \lim_{x\to\infty} \frac{1}{1x+\frac{1}{x}} = 0$$

* 2, Určete monotonii a lokální extrémy

a, $y = \frac{x}{2} - \operatorname{arctg} x$

$$y' = \frac{1}{2} - \frac{1}{1+x^2} = \frac{1+x^2-2}{2(1+x^2)} = \frac{x^2-1}{2(x^2+1)} \qquad D(f) = \mathbb{R}$$

$$\frac{x^2-1}{2(x^2+1)} = 0$$



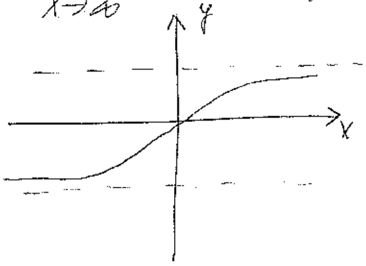
$$(x-1)(x+1) = 0$$

$$x = \pm 1$$

$$f(-1) = \frac{-1}{2} - \operatorname{arctg}(-1) = -\frac{1}{2} + \frac{\pi}{4} \quad \text{MAX.}$$

$$f(1) = \frac{1}{2} - \operatorname{arctg} 1 = \frac{1}{2} - \frac{\pi}{4} \quad \text{MIN.}$$

$$\lim_{x\to\infty} \left(\frac{x}{2} - \operatorname{arctg} x\right) = \infty - \frac{\pi}{2} = \infty$$



$$\lim_{x\to-\infty} \left(\frac{x}{2} - \operatorname{arctg} x\right) = -\infty + \frac{\pi}{2} = -\infty$$

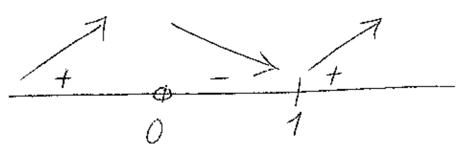
$\pm\infty$ nejsou extrémy

b, $y = x e^{\frac{1}{x}}$

$D(f) = \mathbb{R} \setminus \{0\}$

$$y' = 1 \cdot e^{\frac{1}{x}} + x e^{\frac{1}{x}} \cdot x^{-2}(-1) = e^{\frac{1}{x}}\left(1 - \frac{x}{x^2}\right) = e^{\frac{1}{x}}\left(1 - \frac{1}{x}\right) = e^{\frac{1}{x}} \frac{x-1}{x}$$

$$e^{\frac{1}{x}} \frac{x-1}{x} = 0$$



$$\frac{x-1}{x} = 0 \quad \vee \quad e^{\frac{1}{x}} = 0$$

$$x = 1$$

$$f(1) = 1 \cdot e^1 = e \quad \text{MIN.}$$

$$\lim_{x\to\infty} x e^{\frac{1}{x}} = \infty \cdot e^0 = \infty \qquad \lim_{x\to 0^+} x e^{\frac{1}{x}} = |0 \cdot e^{\infty}| = \lim_{x\to 0^+} \frac{e^{\frac{1}{x}}}{\frac{1}{x}} = \left|\frac{\infty}{\infty}\right| =$$

$$\lim_{x\to-\infty} x e^{\frac{1}{x}} = -\infty \cdot e^0 = -\infty \qquad = \lim_{x\to 0^+} \frac{-\frac{1}{x^2} e^{\frac{1}{x}}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} e^{\frac{1}{x}} = \infty$$

$$\lim_{x\to 0^-} x e^{\frac{1}{x}} = |0\cdot e^{-\infty}| = \lim_{x\to 0^-} \frac{e^{\frac{1}{x}}}{\frac{1}{x}} = \left|\frac{-\infty}{-\infty}\right| = \lim_{x\to 0^-} \frac{-\frac{1}{x^2} e^{\frac{1}{x}}}{-\frac{1}{x^2}} =$$

$$= \lim_{x\to 0^-} e^{\frac{1}{x}} = e^{-\infty} = 0$$

$\pm\infty, 0$ nejsou extrémy

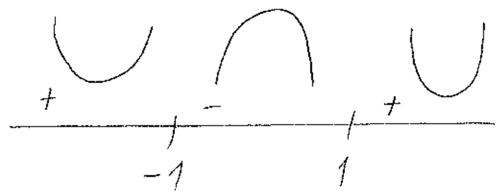
3, Určete konvexnost, konkávnost a inflexní body

a) $y = x^4 - 6x^2 - 2x + 5$

$D(f) = \mathbb{R}$

$y' = 4x^3 - 12x - 2$

$y'' = 12x^2 - 12$



inflexní body: ± 1

$12x^2 - 12 = 0$

$12(x^2-1) = 0$

$(x+1)(x-1) = 0$

$x = \pm 1$

b) $y = \operatorname{arctg} \frac{x}{x+1}$

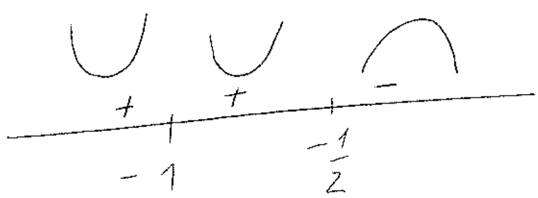
$D(f) = \mathbb{R} \setminus \{-1\}$

$$y' = \frac{1}{1+\frac{x^2}{(x+1)^2}} \cdot \frac{1\cdot(x+1) - x\cdot 1}{(x+1)^2} = \frac{1}{\frac{x^2+2x+1+x^2}{(x+1)^2}} \cdot \frac{x+1-x}{(x+1)^2} = \frac{(x+1)^2}{2x^2+2x+1} \cdot \frac{1}{(x+1)^2} =$$

$$= \frac{1}{2x^2+2x+1}$$

$$y'' = (-1)\cdot(2x^2+2x+1)^{-2}\cdot(4x+2) = \frac{-2(2x+1)}{(2x^2+2x+1)^2}$$

$$\frac{-2(2x+1)}{(2x^2+2x+1)^2} = 0$$



$-2(2x+1) = 0$

$2x+1 = 0$

$x = -\frac{1}{2}$

inflexní body: $-\frac{1}{2}$

4) Určete asymptoty

a) $y = \dfrac{x^3}{x^2-x-2}$ $\qquad x^2-x-2=0$

$D = (-1)^2 - 4\cdot 1\cdot(-2) = 1+8 = 9$

$D(f) = \mathbb{R} \setminus \{-1; 2\}$ $\qquad x_{1,2} = \dfrac{1 \pm 3}{2} = \left\langle \begin{array}{l} 2 \\ -1 \end{array}\right.$

$$\lim_{x \to 2^\pm} \frac{x^3}{\underbrace{x^2-x-2}_{=(x-2)(x+1)}} = \frac{8}{0^\pm \cdot 3} = \pm\infty$$

$$\lim_{x \to -1^\pm} \frac{x^3}{(x-2)(x+1)} = \frac{-1}{-3\cdot 0^\pm} = \pm\infty$$

$\left.\begin{array}{l} x=-1 \\ x=2 \end{array}\right\}$ asymptoty bez směrnice

$y = ax + b$

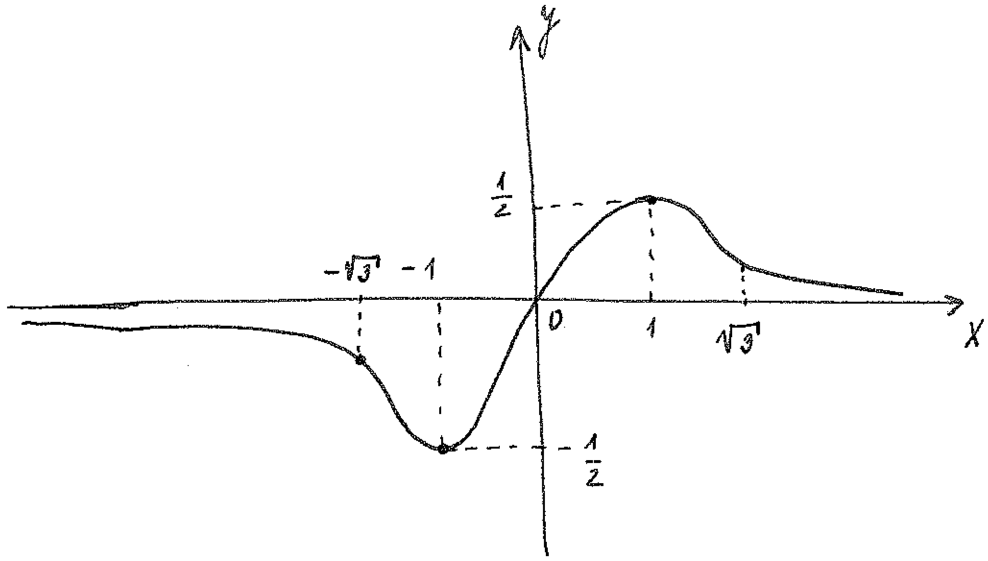
$$a = \lim_{x \to \pm\infty} \frac{x^2}{x^2-x-2} = \lim_{x \to \pm\infty} \frac{1}{1 - \underbrace{\frac{1}{x}}_{\to 0} - \underbrace{\frac{2}{x^2}}_{\to 0}} = 1$$

$$b = \lim_{x \to \pm\infty} \frac{x^3}{x^2-x-2} - 1\cdot x = \lim_{x \to \pm\infty} \frac{x^3 - x^3 + x^2 + 2x}{x^2-x-2} = \lim_{x \to \pm\infty} \frac{x^2+2x}{x^2-x-2} =$$

$$= \lim_{x \to \pm\infty} \frac{1 + \overbrace{\frac{2}{x}}^{\to 0}}{1 - \underbrace{\frac{1}{x}}_{\to 0} - \underbrace{\frac{2}{x^2}}_{\to 0}} = 1$$

$y = x + 1$ asymptota se směrnicí

$$\lim_{x \to -\infty} \frac{x}{x^2+1} = \lim_{x \to -\infty} \frac{1}{x + \frac{1}{x}} = 0$$



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Diferenciál f-ce

$$dx = x - x_0$$

$$dy = f'(x_0)\,dx \rightarrow y - y_0 = dy$$

$$dy = f'(x_0)(x - x_0)$$

tečna : $y = \underbrace{f(x_0)}_{=y_0} + \underbrace{f'(x_0)(x-x_0)}_{=dy}$

$y - y_0 = dy \longrightarrow$ změna funkčních hodnot na tečně

Rozdílem $x - x_0$ se přiblížíme k x_0 a $f(x)$, tzn. pomocí tečny zjistíme přibližnou hodnotu f-ce v nějakém bodě

$$y = y_0 + dy = y_0 + f'(x_0)(x - x_0)$$

Př.: $y = 2x - x^3 \Rightarrow dy = (2 - 3x^2)\,dx$

2, Pomocí diferenciálu přibližně určete hodnotu $(2,1)^{10}$

$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$

$f(x) = x^n = x^{10}$

$x = 2,1$

$x_0 = 2 \ , \ f(x_0) = f(2) = 2^{10} = 1024$

$f'(x) = 10x^9 \ , \ f'(x_0) = f'(2) = 10 \cdot 2^9 = 10 \cdot 512 = 5120$

$f(x) = f(2,1) \approx 1024 + 5120 \cdot (2,1 - 2) = 1024 + 5120 \cdot 0,1 = \underline{\underline{1536}}$

$(2,1)^{10} = 1667,9888$

3, Pomocí diferenciálu přibližně určete hodnotu $e^{0,05}$

$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$

$f(x) = e^x$

$x = 0,05$

$x_0 = 0 \ , \ f(x_0) = e^0 = 1$

$f'(x) = e^x \ , \ f'(x_0) = e^0 = 1$

$f(x) = f(0,05) = 1 + 1 \cdot (0,05 - 0) = \underline{\underline{1,05}}$

$e^{0,05} = 1,05127$

Taylorův polynom

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 +$$

$$+ \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(x), \text{ kde } R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1},$$

$$\xi \in (x, x_0)$$

Pokud položíme $x_0 = 0$, získáme Mclaurinův polynom.

4, Určete Taylorův polynom 3. stupně f-ce $f(x) = \frac{x-1}{x+1}$ se středem v bodě $x_0 = 1$.

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2 + \frac{1}{6} f'''(x_0)(x-x_0)^3 +$$

$$+ \frac{1}{24} f^{(4)}(\xi)(x-x_0)^4, \quad \xi \in (x, 1)$$

$$f(x_0) = \frac{1-1}{1+1} = 0$$

$$f'(x) = \frac{x+1-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2} \qquad f'(x_0) = f'(1) = \frac{2}{4} = \frac{1}{2}$$

$$f''(x) =$$

$$\left(2 \cdot (x+1)^{-2}\right)' = 2 \cdot (-2)(x+1)^{-3} = -\frac{4}{(x+1)^3} \qquad f''(1) = -\frac{4}{8} = -\frac{1}{2}$$

$$f'''(x) = \left[-4(x+1)^{-3}\right]' = -4 \cdot (-3)(x+1)^{-4} = \frac{12}{(x+1)^4} \qquad f'''(1) = \frac{12}{2^4} = \frac{3}{4}$$

$$f''''(x) = \left[12(x+1)^{-4}\right]' = 12(-4)(x+1)^{-5} = -\frac{48}{(x+1)^5} \qquad f^{(4)}(\xi) = -\frac{48}{(\xi+1)^5}$$

$$f(x) = \frac{x-1}{x+1} = 0 + \frac{1}{2}(x-1) + \frac{1}{2} \cdot \left(-\frac{1}{2}\right)(x-1)^2 + \frac{1}{6} \cdot \frac{3}{4}(x-1)^3 + \frac{1}{24} \cdot \frac{-48}{(\xi+1)^5}(x-1)^4 =$$

$$= \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{8}(x-1)^3 - 2\frac{(x-1)^4}{(\xi+1)^5}$$

5) Určete Maclaurinův polynom stupně n f-ce $f(x)=e^x$

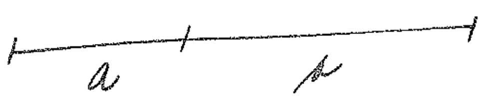
$$x_0 = 0$$

$$f(x) = f(x_0) + f'(x_0)x + \frac{1}{2}f''(x_0)x^2 + \ldots + \frac{1}{n!}f^{(n)}(x_0)x^n$$

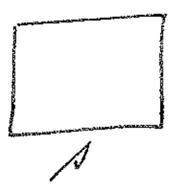
$$f(x_0) = f(0) = e^0 = 1$$
$$f'(x_0) = f'(0) = e^0 = 1 = f''(x_0) = f'''(x_0) = \ldots = f^{(n)}(x_0)$$

$$f(x) = e^x \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \ldots + \frac{1}{n!}x^n =$$
$$= \sum_{i=0}^{n} \frac{1}{i!}x^i$$

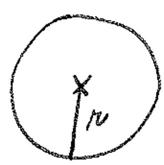
6) Drát délky 4 m rozdělte na dvě části, ze kterých vyrobíte obvod čtverce a kruhu tak, aby součet ploch obou útvarů byl co nejmenší! Určete tento součet.



$a + b = 4\,m$



$a = 4s$



$b = 2\pi r$

$$a + b = 4 \Rightarrow 4s + 2\pi r = 4$$
$$s = 1 - \frac{1}{2}\pi r$$

$$S = s^2 + \pi r^2 = \left(1 - \frac{1}{2}\pi r\right)^2 + \pi r^2 = 1 - \pi r + \frac{1}{4}\pi^2 r^2 + \pi r^2$$

$$S' = -\pi + \frac{1}{4}\pi^2 \cdot 2r + 2\pi r = 0 \Leftrightarrow r\left(\frac{1}{2}\pi^2 + 2\pi\right) = \pi$$

$$r = \frac{\pi}{\pi\left(\frac{1}{2}\pi + 2\right)} = \frac{1}{\frac{1}{2}\pi + 2} = \frac{2}{\pi + 4}$$

$$S'' = \frac{1}{2}\pi^2 + 2\pi$$

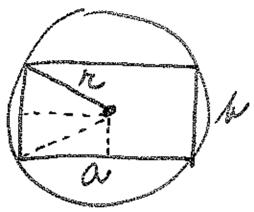
$$S''\left(\frac{2}{\pi+4}\right) = \frac{1}{2}\pi^2 + 2\pi > 0 \Rightarrow \frac{2}{\pi+4} \text{ je minimum}$$

$$r = \frac{2}{\pi+4} \Rightarrow \underline{\underline{b}} = 2\pi \cdot \frac{2}{\pi+4} = \underline{\underline{\frac{4\pi}{\pi+4}}}$$

$$s = 1 - \frac{1}{2}\pi r = 1 - \frac{1}{2}\pi \frac{2}{\pi+4} = 1 - \frac{\pi}{\pi+4} = \frac{\pi+4-\pi}{\pi+4} = \frac{4}{\pi+4} \Rightarrow \underline{\underline{a}} = 4 \cdot \frac{4}{\pi+4} = \underline{\underline{\frac{16}{\pi+4}}}$$

$$S = \left(\frac{4}{\pi+4}\right)^2 + \pi \cdot \left(\frac{2}{\pi+4}\right)^2 = \frac{16}{(\pi+4)^2} + \frac{4\pi}{(\pi+4)^2} = \frac{4(4+\pi)}{(\pi+4)^2} = \underline{\underline{\frac{4}{\pi+4}\ m^2}}$$

7) Do kruhu o poloměru r vepište obdélník s co největším obsahem. Určete velikost jeho stran a, b.



Pythagorova věta: $r^2 = \left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2$

$$4r^2 = a^2 + b^2$$

$$a = \sqrt{4r^2 - b^2}$$

$S = a \cdot b$ má být maximální

$$S = \sqrt{4r^2 - b^2} \cdot b$$

$$S' = \frac{1}{2} \frac{-2b}{\sqrt{4r^2-b^2}} \cdot b + \sqrt{4r^2-b^2} = \frac{-b^2}{\sqrt{4r^2-b^2}} + \sqrt{4r^2-b^2}$$

$$S' = 0 \Leftrightarrow \sqrt{4r^2-b^2} = \frac{b^2}{\sqrt{4r^2-b^2}}$$

$$4r^2 - b^2 = \frac{b^4}{4r^2-b^2}$$

$$16r^4 - 8r^2b^2 + b^4 = b^4$$

$$16r^4 = 8r^2b^2$$

$$\frac{16r^4}{8r^2} = b^2$$

$$2r^2 = b^2$$

$$b = \sqrt{2r^2}$$

$$\underline{\underline{b = \sqrt{2}\,r}}$$

$$S'' = \frac{-2b\sqrt{4r^2-b^2} + b^2\,\frac{1}{2}\,\frac{-2b}{\sqrt{4r^2-b^2}}}{4r^2-b^2} + \frac{1}{2}\,\frac{-2b}{\sqrt{4r^2-b^2}} =$$

$$= \frac{\frac{-2b(4r^2-b^2) - b^3}{\sqrt{4r^2-b^2}}}{4r^2-b^2} - \frac{b}{\sqrt{4r^2-b^2}} = \frac{-8br^2 + 2b^3 - b^3}{(4r^2-b^2)\sqrt{4r^2-b^2}} -$$

$$- \frac{b(4r^2-b^2)}{(4r^2-b^2)\sqrt{4r^2-b^2}} = \frac{8br^2 + b^3 - 4br^2 + b^3}{(4r^2-b^2)\sqrt{4r^2-b^2}} = \frac{4br^2 + b^3}{(4r^2-b^2)\sqrt{4r^2-b^2}}$$

$$S''(\sqrt{2}\,r) = \cancel{8\cdot\sqrt{2}\,r\,r^2 + 2\sqrt{2}\,r^3 - 4\sqrt{2}\,r\,r^2 +} \frac{4\cdot\sqrt{2}\,r^3 + 2\sqrt{2}\,r^3}{(4r^2-2r^2)\sqrt{4r^2-2r^2}} =$$

$$= \frac{6\sqrt{2}\,r^3}{2r^2\sqrt{2r^2}} = \frac{6\sqrt{2}\,r^3}{2r^2\sqrt{2}\,r} = \frac{6\sqrt{2}\,r^3}{2\sqrt{2}\,r^3} = 3 > 0 \Rightarrow$$

$$\underline{\underline{a}} = \sqrt{4r^2-b^2} = \sqrt{4r^2-2r^2} = \sqrt{2r^2} = \underline{\underline{\sqrt{2}\,r}}$$

$a = b \Rightarrow$ čtverec