CVIČENÍ 7

Integrace: primitivní f-ce F k f, pokud platí $F' = f$.

Pak značíme $F = \int f$.

Neurčitý integrál: množina všech primitivních fcí k f.

Je-li f spojitá, pak k ní existuje primitivní f-ce.

Základní integrály:

$$\int k \, dx = kx + c$$

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\int \frac{1}{x} dx = \ln|x|$$

$$\int a^x dx = \frac{a^x}{\ln a}$$

$$\int e^x dx = e^x$$

$$\int \frac{dx}{\sqrt{x^2 \pm k}} = \ln\left|x + \sqrt{x^2 \pm k}\right|$$

$$\int \sin x \, dx = -\cos x$$

$$\int \cos x \, dx = \sin x$$

$$\int \frac{1}{1+x^2} dx = \operatorname{arctg} x = -\operatorname{arccotg} x$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x = -\arccos x$$

$$\int \frac{1}{\sin^2 x} dx = -\operatorname{cotg} x$$

$$\int \frac{1}{\cos^2 x} dx = \operatorname{tg} x$$

Základní vlastnosti:

$$\int f \pm g = \int f \pm \int g$$

$$\int k \cdot f = k \int f$$

$$\int f(ax+b) = \frac{F(ax+b)}{a}$$

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)|$$

$$\int e^{kx} dx = \frac{e^{kx}}{k}$$

Základní metody integrování:

- per partes $\int u' \cdot v = u \cdot v - \int u \cdot v'$

- substituce $f(y) = f(x)$

$$f'(y)\,dy = f'(x)\,dx$$

$$\int_a^b f(x)\,dy \longrightarrow \int_c^d f(y)\,dy$$

1) $\int \frac{1}{\sqrt[3]{x}}\,dx = \int x^{-\frac{1}{3}}\,dx = \frac{x^{\frac{2}{3}}}{\frac{2}{3}} + c = \underline{\underline{\frac{3}{2} x^{\frac{2}{3}} + c}}$

2) $\int e^{-x}\,dx = \left| \begin{matrix} -x = t \\ -dx = dt \end{matrix} \right| = \int e^{t}(-dt) = -e^{t} + c = \underline{\underline{-e^{-x} + c}}$

3) $\int \frac{dx}{x^2+3} = \int \frac{dx}{3\left(\frac{x^2}{3}+1\right)} = \frac{1}{3} \int \frac{dx}{\left(\frac{x}{\sqrt{3}}\right)^2 + 1} = \left| \begin{matrix} \frac{x}{\sqrt{3}} = t \\ x = \sqrt{3}\,t \\ dx = \sqrt{3}\,dt \end{matrix} \right| =$

$= \frac{1}{3} \int \frac{\sqrt{3}\,dt}{t^2+1} = \frac{\sqrt{3}}{3} \int \frac{dt}{t^2+1} = \frac{\sqrt{3}}{3} \operatorname{arctg} t + c = \underline{\underline{\frac{\sqrt{3}}{3} \operatorname{arctg} \frac{x}{\sqrt{3}} + c}}$

4) $\int \frac{1}{\sqrt{4-x^2}}\,dx = \int \frac{dx}{\sqrt{4\left(1-\frac{x^2}{4}\right)}} = \int \frac{dx}{2\sqrt{\left(1-\left(\frac{x}{2}\right)^2\right)}} = \left| \begin{matrix} \frac{x}{2} = t \\ \frac{1}{2}dx = dt \end{matrix} \right| =$

$= \frac{1}{2} \int \frac{2\,dt}{\sqrt{1-t^2}} = \frac{2}{2} \arcsin t + c = \underline{\underline{\arcsin \frac{x}{2} + c}}$

5) $\int \frac{3x^2+1}{x^3+x+1}\,dx = \left| (x^3+x+1)' = 3x^2+1 \right| = \ln|x^3+x+1| + c$

6) $\int \operatorname{tg}^2(au)\,du = \int \frac{\sin^2(au)}{\cos^2(au)}\,du = \left| \begin{matrix} au = t \\ a\,du = dt \end{matrix} \right| =$

$= \int \frac{\sin^2 t}{\cos^2 t} \cdot \frac{1}{a}\,dt = \frac{1}{a} \int \frac{1-\cos^2 t}{\cos^2 t}\,dt = \frac{1}{a} \int \frac{1}{\cos^2 t}\,dt -$

$- \frac{1}{a} \int 1\,dt = \frac{1}{a} \operatorname{tg} t - \frac{1}{a} t + c = \frac{1}{a} \operatorname{tg}(au) - \frac{1}{a} au + c = \underline{\frac{1}{a} \operatorname{tg}(au) - u + c}$

7) $\int tg(bx)\,dx = \left| \begin{matrix} bx = t \\ b\,dx = dt \end{matrix} \right| = \int tg\, t \cdot \frac{1}{b} dt = \frac{1}{b} \int \frac{\sin t}{\cos t} dt =$

$= -\frac{1}{b} \int \frac{-\sin t}{\cos t} dt = \left| (\cos t)' = -\sin t \right| = -\frac{1}{b} \ln|\cos t| + c =$

$= \underline{\underline{-\frac{1}{b} \ln|\cos(bx)| + c}}$

8) $\int (x^2+1) e^{-x} dx = \left| \begin{matrix} u = x^2+1 & v' = e^{-x} \\ u' = 2x & v = -e^{-x} \end{matrix} \right| = -e^{-x}(x^2+1) - \int 2x \cdot (-e^{-x}) dx =$

$= -e^{-x}(x^2+1) + 2 \int x e^{-x} dx = \left| \begin{matrix} u = x & v' = e^{-x} \\ u' = 1 & v = -e^{-x} \end{matrix} \right| = -e^{-x}(x^2+1) +$

$+ 2 \left[-e^{x} \cdot x - \int 1 \cdot (-e^{-x}) dx \right] = -e^{-x}(x^2+1) - 2xe^{-x} + 2 \int e^{-x} dx =$

$= \underline{\underline{-e^{-x}(x^2+1) - 2xe^{-x} + 2(-e^{-x}) + c}}$

9) $\int (2x-1) \ln x \, dx = \left| \begin{matrix} u = \ln x & v' = 2x-1 \\ u' = \frac{1}{x} & v = x^2 - x \end{matrix} \right| = \ln x \cdot (x^2 - x) -$

$- \int \frac{1}{x} \cdot (x^2 - x) dx = (x^2 - x) \ln x - \int (x-1) dx = \underline{\underline{(x^2 - x) \ln x - \frac{x^2}{2} + x + c}}$

10) $\int arctg\, x \, dx = \left| \begin{matrix} u = arctg\, x & v' = 1 \\ u' = \frac{1}{1+x^2} & v = x \end{matrix} \right| = x\, arctg\, x - \int \frac{x}{1+x^2} dx =$

$= x\, arctg\, x - \frac{1}{2} \int \frac{2x}{1+x^2} dx = \left| (1+x^2)' = 2x \right| = \underline{\underline{x\, arctg\, x - \frac{1}{2} \ln|1+x^2| + c}}$

11) $\int e^x \sin x \, dx = \left| \begin{matrix} u = e^x & v' = \sin x \\ u' = e^x & v = -\cos x \end{matrix} \right| = -e^x \cos x + \int e^x \cos x \, dx =$

$= \left| \begin{matrix} u = e^x & v' = \cos x \\ u' = e^x & v = \sin x \end{matrix} \right| = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$

$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$

$2 \int e^x \sin x \, dx = -e^x \cos x + e^x \sin x$

$\int e^x \sin x \, dx = \underline{\underline{\frac{e^x(\sin x - \cos x)}{2} + c}}$

12) $\int \cos^2 x\,dx = \left| \begin{matrix} u=\cos x & v'=\cos x \\ u'=-\sin x & v=\sin x \end{matrix} \right| = \cos x \sin x + \int \sin^2 x\,dx =$

$= \cos x \sin x + \int (1-\cos^2 x)\,dx = \cos x \sin x + x - \int \cos^2 x\,dx$

$\int \cos^2 x\,dx = \cos x \sin x + x - \int \cos^2 x\,dx$

$2\int \cos^2 x\,dx = \cos x \sin x + x$

$\int \cos^2 x\,dx = \underline{\underline{\frac{1}{2}(\cos x \sin x + x) + C}}$

13) $\int \frac{x}{\cos^2 x}\,dx = \left| \begin{matrix} u=x & v'=\frac{1}{\cos^2 x} \\ u'=1 & v=\operatorname{tg} x \end{matrix} \right| = x\operatorname{tg} x - \int \operatorname{tg} x\,dx =$

$= x\operatorname{tg} x + \int \frac{-\sin x}{\cos x}\,dx = \underline{\underline{x\operatorname{tg} x + \ln|\cos x| + C}}$

14) $\int \ln x\,dx = \left| \begin{matrix} u=\ln x & v'=1 \\ u'=\frac{1}{x} & v=x \end{matrix} \right| = x\ln x - \int 1\,dx = \underline{\underline{x\ln x - x + C}}$

15) $\int \frac{\cos x}{\sqrt{1+\sin^2 x}}\,dx = \left| \begin{matrix} t=\sin x \\ dt=\cos x\,dx \end{matrix} \right| = \int \frac{dt}{\sqrt{1+t^2}} = \left| \begin{matrix} a = t+\sqrt{1+t^2} \\ da = \left(1+\frac{2t}{2\sqrt{1+t^2}}\right)dt = \frac{\sqrt{t^2+1}+t}{\sqrt{1+t^2}}dt \\ \frac{da}{\sqrt{1+t^2}+t} = \frac{dt}{\sqrt{1+t^2}} \end{matrix} \right| =$

$= \int \frac{da}{a} = \ln|a| + C = \underline{\underline{\ln\left|\sin x + \sqrt{1+\sin^2 x}\right| + C}}$

16) $\int \frac{(1+\ln x)^4}{x}\,dx = \left| \begin{matrix} t=\ln x \\ dt=\frac{1}{x}dx \end{matrix} \right| = \int (1+t)^4\,dt = \frac{(1+t)^5}{5} + C = \underline{\underline{\frac{(1+\ln x)^5}{5} + C}}$

17) $\int \sin x \cdot \cos^5 x\,dx = \left| \begin{matrix} t=\cos x \\ dt=-\sin x\,dx \end{matrix} \right| = -\int t^5\,dt = -\frac{t^6}{6} + C = \underline{\underline{-\frac{\cos^6 x}{6} + C}}$

18) $\int (4-7x)^{10}\,dx = \left| \begin{matrix} t=4-7x \\ dt=-7dx \end{matrix} \right| = -\frac{1}{7}\int t^{10}\,dt = -\frac{1}{7}\frac{t^{11}}{11} + C = \underline{\underline{-\frac{1}{7}\frac{(4-7x)^{11}}{11} + C}}$

19) $\int \sqrt{2x-5}\,dx = \left| \begin{matrix} t=2x-5 \\ dt=2dx \end{matrix} \right| = \frac{1}{2}\int \sqrt{t}\,dt = \frac{1}{2}\int t^{\frac{1}{2}}dt = \frac{1}{2}\cdot\frac{2}{3}t^{\frac{3}{2}} + C =$

$= \underline{\underline{\frac{1}{3}\sqrt{(2x-5)^3} + C}}$

20) $\int \frac{\cos x}{(2+\sin x)^2}dx = \left| \begin{matrix} t=2+\sin x \\ dt=\cos x\,dx \end{matrix} \right| = \int \frac{dt}{t^2} = \frac{t^{-1}}{-1}+C = \underline{\underline{-\frac{1}{2+\sin x}+C}}$

21) $\int e^{\sqrt{x}}dx = \left| \begin{matrix} t=\sqrt{x} \\ t^2=x \\ 2t\,dt=dx \end{matrix} \right| = 2\int e^t dt = \left| \begin{matrix} u=t & v'=e^t \\ u'=1 & v=e^t \end{matrix} \right| = 2[te^t -$

$- \int e^t dt] = 2te^t - 2e^t + C = \underline{\underline{2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C}}$

22) $\int x \cdot \arcsin x^2 dx = \left| \begin{matrix} t=x^2 \\ dt=2x\,dx \end{matrix} \right| = \frac{1}{2}\int \arcsin t\,dt = \left| \begin{matrix} u=\arcsin t \\ u'=\frac{1}{\sqrt{1-t^2}} \end{matrix} \right.$

$\left. \begin{matrix} v'=1 \\ v=t \end{matrix} \right| = \frac{1}{2}\left[t\arcsin t - \int \frac{t}{\sqrt{1-t^2}}dt\right] = \left| \begin{matrix} a=1-t^2 \\ da=-2t\,dt \end{matrix} \right| =$

$= \frac{1}{2}t\arcsin t - \frac{1}{2}\int \frac{1}{\sqrt{a}}\left(-\frac{1}{2}\right)da = \frac{1}{2}t\arcsin t + \frac{1}{4}\int \frac{1}{a^{\frac{1}{2}}}da =$

$= \frac{1}{2}t\arcsin t + \frac{1}{4}2\cdot\sqrt{a} + C = \underline{\underline{\frac{1}{2}x^2\arcsin x^2 + \frac{1}{2}(1-x^4)^{\frac{1}{2}} + C}}$

23) $\int \sqrt{1-x^2}\,dx = \left| \begin{matrix} x=\sin t \\ dx=\cos t\,dt \end{matrix} \right| = \int \sqrt{1-\sin^2 t}\cdot\cos t\,dt =$

$= \int \cos^2 t\,dt = \left| \begin{matrix} u=\cos t & v'=\cos t \\ u'=-\sin t & v=\sin t \end{matrix} \right| = \cos t\sin t + \int \sin^2 t\,dt =$

$= \cos t\sin t + \int 1dt - \int \cos^2 t\,dt = *$

$\int \cos^2 t\,dt = \cos t\sin t + t - \int \cos^2 t\,dt$

$\int \cos^2 t\,dt = \frac{1}{2}(\cos t\sin t + t) + C$

$* = \frac{1}{2}\sqrt{1-\sin^2 t}\,\sin t + \frac{t}{2} + C = \underline{\underline{\frac{1}{2}\sqrt{1-x^2}\cdot x + \frac{1}{2}\arcsin x + C}}$

24) $\int \frac{3x+7}{x^2-4x+15}dx = \frac{3}{2}\int \frac{2x-4}{x^2-4x+15}dx + 13\int \frac{dx}{x^2-4x+15} =$

$$= \frac{3}{2}\ln|x^2-4x+15| + 13\int\frac{dx}{(x-2)^2+11} = \frac{3}{2}\ln|x^2-4x+15| +$$

$$+ \frac{13}{11}\int\frac{dx}{\left(\frac{x-2}{\sqrt{11}}\right)^2+1} = \left|\begin{array}{l} t = \frac{x-2}{\sqrt{11}} \\ dt = \frac{1}{\sqrt{11}}dx \end{array}\right| = \frac{3}{2}\ln|x^2-4x+15| +$$

$$+ \frac{13}{11}\int\frac{\sqrt{11}}{t^2+1}dt = \underline{\underline{\frac{3}{2}\ln|x^2-4x+15| + \frac{13\sqrt{11}}{11}\operatorname{arctg}\frac{x-2}{\sqrt{11}} + C}}$$

25) $\int\frac{x^3+1}{x(x-1)^3}dx = *$

$$\frac{x^3+1}{x(x-1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$$

$$x^3+1 = A(x-1)^3 + Bx(x-1)^2 + Cx(x-1) + Dx$$

$$x^3+1 = Ax^3 - 3Ax^2 + 3Ax - A + Bx^3 - 2Bx^2 + Bx + Cx^2 - Cx + Dx$$

$A+B=1$

$-3A-2B+C=0$

$3A+B-C+D=0$

$-A=1$

$A=-1$

$-3+B-C+D=0$

$3-2B+C=0$

$-1+B=1$

$B=2$

$-3+2-C+D=0$

$3-4+C=0$

$C=1$

$D=2$

$$* = -\int\frac{dx}{x} + 2\int\frac{dx}{x-1} + \int\frac{dx}{(x-1)^2} + 2\int\frac{dx}{(x-1)^3} = \left|\begin{array}{l} t=x-1 \\ dt=dx \end{array}\right| = -\ln|x| +$$

$$+ 2\ln|x-1| + \int\frac{dt}{t^2} + 2\int\frac{dt}{t^3} = -\ln|x| + 2\ln|x-1| - \frac{1}{t} + 2\cdot\frac{t^{-2}}{-2} + C =$$

$$= \underline{\underline{-\ln|x| + 2\ln|x-1| - \frac{1}{x-1} - \frac{1}{(x-1)^2} + C}}$$