CVIČENÍ 9

Určete součet řad

1) $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

$$\frac{1}{n^2+n} = \frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$$1 = A(n+1) + Bn$$

$$1 = (A+B)n + A$$

$$A = 1$$

$$A + B = 0$$

$$A = -B$$

$$B = -1$$

$$\frac{1}{n^2+n} = \frac{1}{n} + \frac{-1}{n+1}$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim_{n\to\infty}\left(\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \dots\right) = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1 - 0 = \underline{\underline{1}}$$

2) $\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$

$$\frac{1}{n(n+3)} = \frac{A}{n} + \frac{B}{n+3}$$

$$1 = An + 3A + Bn$$

$$1 = (A+B)n + 3A$$

$$A + B = 0$$

$$3A = 1$$

$$A = \frac{1}{3}$$

$$B = -\frac{1}{3}$$

$$\frac{1}{n(n+3)} = \frac{1}{3n} - \frac{1}{3(n+3)}$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{1}{3n} - \frac{1}{3(n+3)}\right) = \lim_{n\to\infty} \frac{1}{3}\left(\frac{1}{n} - \frac{1}{n+3}\right) = \lim_{n\to\infty} \frac{1}{3}\left(\frac{1}{1} - \frac{1}{4} + \frac{1}{2} - \frac{1}{5} + \frac{1}{3} - \frac{1}{6} + \frac{1}{4} - \frac{1}{7} + \frac{1}{5} - \frac{1}{8} + \dots - \frac{1}{n+3}\right) = \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) = \underline{\frac{11}{18}}$$

3) $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$

$$\frac{1}{4n^2-1} = \frac{A}{2n+1} + \frac{B}{2n-1}$$

$$1 = 2nA - A + 2nB + B$$

$$1 = 2n(A+B) - A + B$$

$$2(A+B) = 0$$

$$-A + B = 1$$

$$B = A+1$$

$$2(A + A + 1) = 0$$

$$2A + 1 = 0$$

$$2A = -1$$

$$A = -\frac{1}{2}$$

$$B = \frac{1}{2}$$

$$\frac{1}{4n^2-1} = -\frac{1}{2(2n+1)} + \frac{1}{2(2n-1)}$$

$$\lim_{n\to\infty} \frac{1}{2}\left(-\frac{1}{2n+1} + \frac{1}{2n-1}\right) = \lim_{n\to\infty} \frac{1}{2}\left[\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \right.$$

$$\left. + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)\right] = \lim_{n\to\infty} \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) = \underline{\underline{\frac{1}{2}}}$$

4) $\sum_{n=1}^{\infty} \frac{3^n + 2^n}{6^n}$

$$\sum_{n=1}^{\infty} \frac{3^n+2^n}{6^n} = \sum_{n=1}^{\infty}\left[\left(\frac{3}{6}\right)^n + \left(\frac{2}{6}\right)^n\right] = \sum_{n=1}^{\infty}\left[\left(\frac{1}{2}\right)^n + \left(\frac{1}{3}\right)^n\right] = \frac{1}{2}\cdot\frac{1}{1-\frac{1}{2}} + \frac{1}{3}\,\frac{1}{1-\frac{1}{3}} =$$

$$= 1 + \frac{1}{2} = \underline{\underline{\frac{3}{2}}}$$

Geometrická řada: $|q| < 1$ $\quad s_n = a_1 \cdot \frac{1}{1-q}$

5) $\sum_{n=1}^{\infty} \frac{2n-1}{2^n} =$

$$\sum_{n=1}^{\infty} \frac{2n-1}{2^n} = \sum_{n=1}^{\infty}\left(\frac{2n}{2^n} - \frac{1}{2^n}\right) = 2\sum_{n=1}^{\infty}\frac{n}{2^n} - \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$$

$\sum_{n=1}^{\infty} \frac{n}{2^n}$:

$$s_n = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \frac{5}{2^5} + \frac{6}{2^6} + \dots + \frac{n}{2^n}$$

$$\frac{s_n}{2} = \frac{1}{4} + \frac{2}{2^3} + \frac{3}{2^4} + \frac{4}{2^5} + \frac{5}{2^6} + \dots + \frac{n-1}{2^n} + \frac{n}{2^{n+1}}$$

$$s_n - \frac{s_n}{2} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots + \frac{1}{2^n} - \frac{n}{2^{n+1}}$$

$$\frac{s_n}{2} = \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n} - \frac{n}{2^{n+1}}$$

$$s_n = \underbrace{1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^{n-1}}}_{\text{geometrická řada} \quad 1\cdot\frac{1}{1-\frac{1}{2}} = 2} - \frac{n}{2^{n+1}}$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(2 - \frac{n}{2^{n+1}}\right) = 2 - 0 = 2$$

$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ je geometrická řada se součtem $\frac{1}{2}\cdot \frac{1}{1-\frac{1}{2}} = 1$

$$2\sum_{n=1}^{\infty} \frac{n}{2^n} - \sum_{n=1}^{\infty} \frac{1}{2^n} = 2\cdot 2 - 1 = \underline{\underline{3}}$$

6, $\sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} + \frac{2}{3^{n-1}}\right) = \sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{2}{3^n}\right) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n + 2\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n =$

$$= 1\cdot \frac{1}{1-\frac{1}{2}} + 2\cdot 1\cdot \frac{1}{1-\frac{1}{3}} = 2 + 2\cdot\frac{3}{2} = 2 + 3 = \underline{\underline{5}}$$

Rozhodněte o konvergenci řad

7, $\sum_{n=1}^{\infty} \ln n$

nutná podm. konvergence $\lim_{n\to\infty} a_n = 0$

$\lim_{n\to\infty} \ln n = \infty \neq 0 \Rightarrow$ diverguje

8, $\sum_{n=1}^{\infty} \frac{1}{\operatorname{arctg} n}$

$\lim_{n\to\infty} \frac{1}{\operatorname{arctg} n} = \frac{1}{\frac{\pi}{2}} \neq 0 \Rightarrow$ diverguje

9, $\sum_{n=1}^{\infty} \frac{n^2}{2n^2+1}$

$\lim_{n\to\infty} \frac{n^2}{2n^2+1} = \lim_{n\to\infty} \frac{n^2}{n^2(2+\frac{1}{n^2})} = \lim_{n\to\infty} \frac{1}{2+0} = \frac{1}{2} \neq 0 \Rightarrow$ diverguje

10, Riešte rovnicu v $\mathbb{R}$:

$$\log x + \log \sqrt{x} + \log \sqrt[4]{x} + \log \sqrt[8]{x} + \ldots = 2$$

$$\sum_{n=0}^{\infty} \log \sqrt[2^n]{x} = 2$$

$$\log \sqrt[2^n]{x} = \log x^{\frac{1}{2^n}} = \frac{1}{2^n} \log x \Rightarrow \sum_{n=0}^{\infty} \log \sqrt[2^n]{x} = \log x \cdot \sum_{n=0}^{\infty} \frac{1}{2^n} =$$

$$= \log x \cdot 1 \cdot \frac{1}{1-\frac{1}{2}} = 2 \log x$$

$$2 \log x = 2$$

$$\log x = 1$$

$$10^1 = x$$

$$\underline{\underline{x = 10}}$$

Pomocou srovnávacieho kritéria rozhodnite o konvergencii řady

11, $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+4)}$

$$\lim_{n \to \infty} \frac{1}{(n+1)(n+4)} = 0 \checkmark$$

srovnávací: $(n+1)(n+4) > n^2$

$$\frac{1}{(n+1)(n+4)} < \frac{1}{n^2}$$

Tím, že $\sum_{n=1}^{\infty} \frac{1}{n^2}$ konverguje a $\sum_{n=1}^{\infty} \frac{1}{n^2} > \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+4)} \Rightarrow$
$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+4)}$ konverguje.

12) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$

$\lim_{n\to\infty} \frac{1}{n(n+1)(n+2)} = 0$

limitní: $\lim_{n\to\infty} \frac{\frac{1}{n(n+1)(n+2)}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n(n+1)(n+2)} = \lim_{n\to\infty} \frac{1}{n+3+\frac{2}{n}} =$

$= 0$

$L < \infty \wedge \sum_{n=1}^{\infty} \frac{1}{n^2}$ konverguje $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$ konverguje

13) $\sum_{n=1}^{\infty} \frac{n^2}{n!}$

$\lim_{n\to\infty} \frac{n^2}{n!} = \lim_{n\to\infty} \frac{n^2}{n(n-1)(n-2)(n-3)\dots 1} = \lim_{n\to\infty} \frac{n}{(n-1)(n-2)\dots 1} =$

$= \lim_{n\to\infty} \frac{1\, n}{n(1-\frac{1}{n})(n-2)\dots 1} = \lim_{n\to\infty} \frac{1}{(1-\frac{1}{n})(n-2)\dots 1} = 0$

$\downarrow$

0

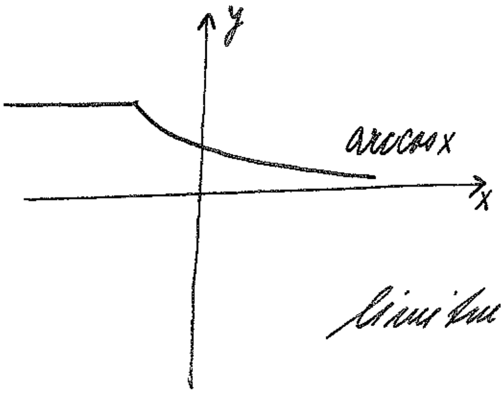
podílové: $\lim_{n\to\infty} \frac{\frac{(n+1)^2}{(n+1)!}}{\frac{n^2}{n!}} = \lim_{n\to\infty} \frac{(n+1)^2}{(n+1)n!} \cdot \frac{n!}{n^2} =$

$= \lim_{n\to\infty} \frac{(n+1)^2}{n^2(n+1)} = \lim_{n\to\infty} \frac{n+1}{n^2} = \lim_{n\to\infty} \frac{n(1+\frac{1}{n})}{n \cdot n} =$

$= \lim_{n\to\infty} \frac{1}{n} = 0$

$L < 1 \Rightarrow$ konverguje

14) $\sum_{n=1}^{\infty} \arccos \frac{n}{n+1}$



Jedná se o řadu s nezápornými členy.

$\lim_{n\to\infty} \arccos \frac{n}{n+1} = 0$ (podle grafu je to zřejmé)

limitní: $\lim_{n\to\infty} \frac{\arccos \frac{n}{n+1}}{\frac{1}{n}} = \left|\frac{0}{0}\right| =$

$= \lim_{n\to\infty} \frac{-\frac{1}{\sqrt{1-(\frac{n}{n+1})^2}} \cdot \frac{n+1-n}{n^2}}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{\sqrt{\frac{2n+1}{(n+1)^2}}} \cdot \frac{n^2}{(n+1)^2} =$

$$= \lim_{n\to\infty} \frac{n+1}{\sqrt{2n+1}} \cdot \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \frac{n^2}{(n+1)\sqrt{2n+1}} = \lim_{n\to\infty} \frac{n^2}{\sqrt{n}\sqrt{2+\frac{1}{n}}\,(n+\frac{1}{n})n} =$$

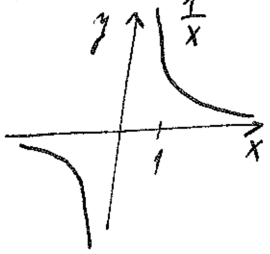
$$= \lim_{n\to\infty} \frac{n}{\sqrt{n}\,(1+\frac{1}{n})\sqrt{2+\frac{1}{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{(1+\frac{1}{n})\sqrt{2+\frac{1}{n}}} = \infty$$

$L > 0 \wedge \sum_{n=1}^{\infty} \frac{1}{n}$ diverguje $\Rightarrow \sum_{n=1}^{\infty} \arccos \frac{n}{n+1}$ diverguje

15) $\sum_{n=1}^{\infty} \frac{1}{n}$

integrálně: $\int_1^{\infty} \frac{1}{x}\,dx = [\ln x]_1^{\infty} = \infty \Rightarrow$ diverguje

Zbývá ověřit, že se jedná o f-ci klesající, nezápornou a spojitou na $(1,\infty)$.



Tyto vlastnosti lze jednoduše vyčíst z grafu f-ce $f(x) = \frac{1}{x}$.

16) $\sum_{n=1}^{\infty} \frac{\sin n}{6^n}$

Jedná se o řadu s libovolnými členy, proto použijeme kritérium absolutní konvergence.

$\left|\frac{\sin n}{6^n}\right| \leq \frac{1}{6^n}$ $\quad \sum_{n=1}^{\infty} \frac{1}{6^n}$ je geometrická řada, je konvergentní, a tudíž dle srovnávacího kritéria je i řada $\sum_{n=1}^{\infty} \left|\frac{\sin n}{6^n}\right|$ konvergentní, což podle kritéria absolutní konvergence znamená, že $\sum_{n=1}^{\infty} \frac{\sin n}{6^n}$ je konvergentní.

17, Vyjádři ve tvaru zlomku číslo $0,2\overline{15}$.

$$0,2\overline{15} = \frac{2}{10} + \left(\frac{15}{10^3} + \frac{15}{10^5} + \frac{15}{10^7} + \frac{15}{10^9} + \dots\right) =$$

$$= \frac{1}{5} + \frac{15}{10^3}\underbrace{\left(1 + \frac{1}{10^2} + \frac{1}{10^5} + \dots\right)}_{\text{geometrická řada}} = \frac{1}{5} + \frac{15}{10^3} \cdot \frac{1}{1 - \frac{1}{10^2}} =$$

$$= \frac{1}{5} + \frac{15}{10 \cdot 99} = \frac{1}{5} + \frac{1}{66} = \underline{\underline{\frac{71}{330}}}$$

18, Rozhodněte o konvergenci řady užitím srovnávacích kritérií.

$$\sum_{n=1}^{\infty} \sin \frac{\pi}{n}$$

limitní: $\lim\limits_{n \to \infty} \dfrac{\sin \frac{\pi}{n}}{\frac{1}{n}} = \left|\dfrac{0}{0}\right| = \lim\limits_{n \to \infty} \dfrac{\pi\left(\frac{1}{n}\right)' \cos \frac{\pi}{n}}{\left(\frac{1}{n}\right)'} =$

$= \lim\limits_{n \to \infty} \pi \cdot \cos \frac{\pi}{n} = \pi > 0 \Rightarrow$ diverguje, ptž. $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverguje

19, Rozhodněte o konvergenci užitím srovnávacích kritérií.

$$\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n^2}\right)$$

limitní: $\lim\limits_{n \to \infty} \dfrac{\ln\left(1 + \frac{1}{n^2}\right)}{\frac{1}{n^2}} = \left|\dfrac{0}{0}\right| = \lim\limits_{n \to \infty} \dfrac{\frac{1}{1 + \frac{1}{n^2}} \cdot \left(\frac{1}{n^2}\right)'}{\left(\frac{1}{n^2}\right)'} =$

$= \lim\limits_{n \to \infty} \dfrac{1}{1 + \frac{1}{n^2}} = 1 < \infty \Rightarrow$ konverguje, ptž. $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ konverguje

20, Rozhodněte o konvergenci užitím srovnávacích kritérií.

$$\sum_{n=1}^{\infty} \frac{n}{\left(3 + \frac{1}{n}\right)^n}$$

odmocninové: $\lim\limits_{n \to \infty} \sqrt[n]{\dfrac{n}{\left(3 + \frac{1}{n}\right)^n}} = \lim\limits_{n \to \infty} \dfrac{\sqrt[n]{n}}{\left(3 + \frac{1}{n}\right)} = \dfrac{1}{3} < 1 \Rightarrow$

řada je konvergentní

21, Rozhodněte o konvergenci užitím vhodného kritéria.

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

integrální: $f(x) = \dfrac{1}{x \ln x}$, $x \in [2; \infty)$

$$f'(x) = \frac{-(\ln x + 1)}{x^2 \ln^2 x}$$

$$-\frac{\ln x + 1}{x^2 \ln^2 x} = 0 \iff \ln x + 1 = 0 \iff \ln x = -1 \iff x = \frac{1}{e}$$

$$-\frac{\ln x + 1}{x^2 \ln^2 x} < 0 \iff x \in R$$

$\ln x + 1 > 0 \iff \ln x > -1 \iff x > \frac{1}{e}$ $\quad$ $\frac{1}{e} < 2 \Rightarrow$ na $[2, \infty)$ je

f-ce $f(x) = \frac{1}{x \ln x}$ nerostoucí, nezáporná a spojitá

$$\int_2^{\infty} \frac{1}{x \ln x}\, dx = \lim_{x \to \infty} \int_2^x \frac{1}{t \ln t}\, dt = \left| \begin{matrix} y = \ln t \\ dy = \frac{1}{t}\, dt \end{matrix} \right| = \lim_{x \to \infty} \int_2^x \frac{1}{y}\, dy =$$

$$= \lim_{x \to \infty} \left[\ln(\ln x) - \ln(\ln 2) \right]_2^x = \infty \Rightarrow \text{diverguje}$$

22, Rozhodněte o konvergenci $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{3n+1}{2n-3}$

$$\frac{3n+1}{2n-3} > 0$$

$$\lim_{n \to \infty} \frac{3n+1}{2n-3} = \frac{3}{2} \neq 0 \Rightarrow \text{diverguje}$$

23, Rozhodněte o konvergenci $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{n - \ln n}$

$$f(x) = \frac{1}{x - \ln x} \qquad f'(x) = -\frac{1 - \frac{1}{x}}{(x - \ln x)^2}$$

$$\frac{\frac{1}{x} - 1}{(x - \ln x)^2} < 0$$

$$\frac{1}{x} - 1 < 0$$

$$\frac{1}{x} < 1$$

$$x > 1$$

Pro $x \in (1;\infty)$ je to f-ce klesající.

$$\lim_{n\to\infty} \frac{1}{n - \ln n} = \frac{1}{\lim\limits_{n\to\infty} (n - \ln n)} = \frac{1}{\lim\limits_{n\to\infty} (\ln e^{n} - \ln n)} =$$

$$= \frac{1}{\lim\limits_{n\to\infty} \left(\ln \frac{e^{n}}{n}\right)} = \left|\frac{1}{\infty}\right| = 0 \Rightarrow \text{řada konverguje}$$

Vyšetřete absolutní konvergenci

24) $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{1}{(2n+1)^3}$

$$\left|(-1)^{n+1} \frac{1}{(2n+1)^3}\right| = \frac{1}{(2n+1)^3}$$

$$(2n+1)^3 > n^3$$

$$\frac{1}{(2n+1)^3} < \frac{1}{n^3}$$

$\sum\limits_{n=1}^{\infty} \frac{1}{n^3}$ je konvergentní, a tedy $\sum\limits_{n=1}^{\infty} \left|(-1)^{n+1} \frac{1}{(2n+1)^3}\right|$ je konvergentní $\Rightarrow$

$\Rightarrow \sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{1}{(2n+1)^3}$ je konvergentní absolutně

25) $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\ln^{n}(n+1)}$

$$\left|(-1)^{n+1} \frac{1}{\ln^{n}(n+1)}\right| = \frac{1}{\ln^{n}(n+1)}$$

$\lim\limits_{n\to\infty} \sqrt[n]{\frac{1}{\ln^{n}(n+1)}} = \lim\limits_{n\to\infty} \frac{1}{\ln(n+1)} = 0 < 1 \Rightarrow$ řada je absolutně konvergentní

26, Určete pro která $x \in \mathbb{R}$ je řada $\sum_{n=1}^{\infty} \frac{x^n}{n}$ absolutně konvergentní, neabsolutně konvergentní a divergentní.

$$x \neq 0: \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{n+1} \frac{n}{x^n}\right| = \lim_{n\to\infty} \frac{n}{n+1} \cdot |x| = |x|$$

$|x| < 1$ je řada absolutně konvergentní

$|x| > 1$ je řada divergentní

$x = 1: \sum \frac{1}{n}$ je řada divergentní

$x = -1: \sum (-1)^n \frac{1}{n}$ je řada konvergentní