Digital Signal Processing

An Overview of Complex Numbers

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The Complex Number System

Example

- Consider throwing a ball straight up into air with an initial velocity (v_i) of 9.8 meters/sec
  Height of ball (h) at any instant of time (t)

  $$h = \frac{-gt^2}{2} + v_i t \longrightarrow t = 1 \pm \sqrt{1 - h/4.9}$$

  E.g., ball reaches $h = 3$ meters twice: $t = 0.38$ (going up), $t = 1.62$ (going down)
- $h = 10 \longrightarrow t = ?$
  - Never in reality
- But $h = 10 \longrightarrow t = 1 + \sqrt{-1.041}$ and $t = 1 - \sqrt{-1.041}$
  - Contain square-root of a negative number
  - Called **complex numbers**

The Complex Number System

- Every complex number is sum of two components
  - A **real part**: an ordinary number
  - An **imaginary part**: square-root of a negative number
- Imaginary part is usually reduced to an ordinary number times $\sqrt{-1}$

Example

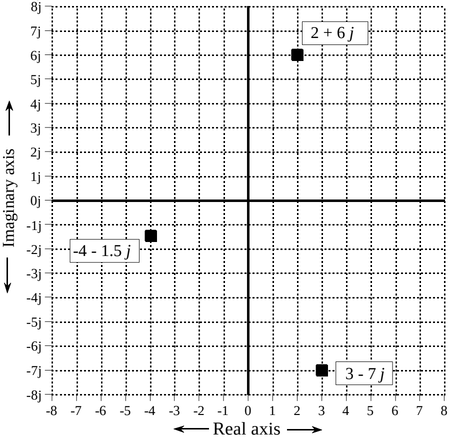
- Consider

$$\begin{aligned} t &= 1 + \sqrt{-1.041} \\ &= 1 + \sqrt{1.041}\sqrt{-1} \\ &= 1 + 1.02\sqrt{-1} \end{aligned}$$

- Real part: 1
  Imaginary part: $1.02\sqrt{-1}$
- Abstract term $\sqrt{-1}$ is given a special symbol: **j** (sometimes **i**)
- Therefore, $t = 1 + 1.02j$

The Complex Number System

- Complex numbers are represented by locations in a two-dimensional display called **complex plane**
  - Horizontal axis = real part of complex number
  - Vertical axis = imaginary part



The Complex Number System

- In equations, a complex number is represented by a single variable

Example

$$A = 2 + 6j$$
$$Re\,A = 2,\; Im\,A = 6$$

- Complex numbers follow the same algebra as ordinary numbers, treating j as a constant
- Addition, subtraction, multiplication, and division of complex numbers

$$(a + bj) + (c + dj) = (a + c) + j(b + d)$$

$$(a + bj) - (c + dj) = (a - c) + j(b - d)$$

$$(a + bj)(c + dj) = (ac - bd) + j(bc + ad)$$
$$\frac{(a + bj)}{(c + dj)} = \left(\frac{ac + bd}{c^2 + d^2}\right) + j\left(\frac{bc - ad}{c^2 + d^2}\right)$$

The Complex Number System

- Complex conjugate
  - A complex number with sign of imaginary part switched

$$Z = a + bj$$
$$Z^* = a - bj$$

- Some properties of complex numbers
  1. Commutative property

$$AB = BA$$

  2. Associative property

$$(A + B) + C = A + (B + C)$$

  3. Distributive property

$$A(B + C) = AB + AC$$

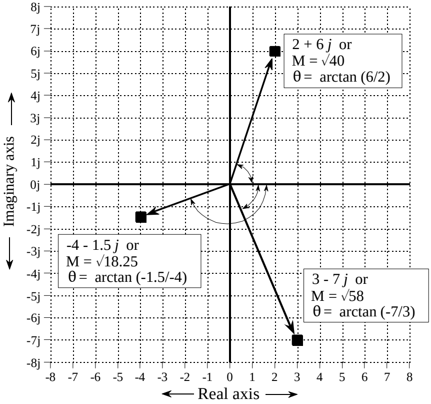
Polar Notation

- Complex numbers can be expressed in two notations
  - Rectangular notation (which was just described)
  - Polar notation
- In polar notation
  - Magnitude
    - Length of vector starting at origin and ending at complex point
  - Phase angle
    - measured between this vector and positive x-axis
- Rectangular-to-polar conversion

$$M = \sqrt{(Re\,A)^2 + (Im\,A)^2}$$

$$\theta = \arctan\left[\frac{Im\,A}{Re\,A}\right]$$

Polar Notation



- Polar-to-rectangular conversion

$$Re\,A = M\cos(\theta)$$
$$Im\,A = M\sin(\theta)$$

- Using above equations

$$a + jb = M(\cos\theta + j\sin\theta)$$

Polar Notation

- Euler's relation

$$e^{jx} = \cos x + j \sin x$$

- Rewriting equation

$$a + jb = M(\cos\theta + j\sin\theta)$$

using Euler's relation results in a **complex exponential**

$$a + jb = Me^{j\theta}$$

- Using exponential polar form makes multiplication and division simple

$$M_1 e^{j\theta_1} M_2 e^{j\theta_2} = M_1 M_2 e^{j(\theta_1+\theta_2)}$$

$$\frac{M_1 e^{j\theta_1}}{M_2 e^{j\theta_2}} = \left[\frac{M_1}{M_2}\right] e^{j(\theta_1-\theta_2)}$$

Polar Notation

- In Euler's relation

$$e^{jx} = \cos(x) + j\sin(x)$$
$$\text{or } e^{-jx} = \cos(x) - j\sin(x)$$

  one complex expression is equal to another complex expression
  - This is not useful
  - Rearranging relations

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$
$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j} = \frac{j(e^{-jx} - e^{jx})}{2}$$

Using Complex Numbers

- Question
  - How to use a mathematics that has no connection with everyday experience?
- Answer
  - Change physical problem into a complex number form
  - Manipulate complex numbers
  - Then change back into a physical answer
- Two ways that physical problems can be represented using complex numbers
  - Substitution
  - Mathematical equivalence

Using Complex Numbers by Substitution

- Substitution
  - Takes two real physical parameters
  - Places one in real part of complex number and one in imaginary part
  - After mathematical operations, complex number is separated into its real and imaginary parts corresponding to physical parameters
- Substitution allows two values to be manipulated as a single complex number

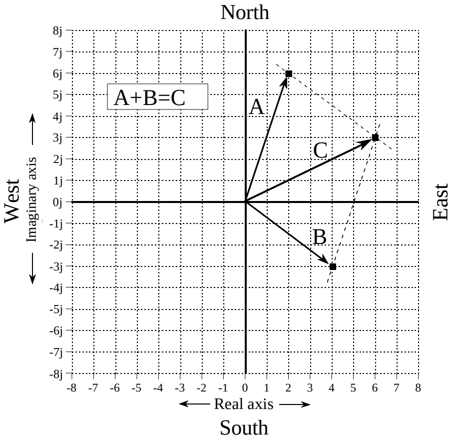
Example

- A boat is pushed in one direction by wind, and in another direction by ocean current
- Resulting force is vector sum of two force vectors
- Use complex numbers
  - Place east/west coordinate into real part of a complex number
  - North/south coordinate into imaginary part
- Substitution allows us to treat each vector as a single complex number

Using Complex Numbers by Substitution

Example (Continued)

- Wind (2 parts to east, 6 parts to north) $\longrightarrow A: 2 + 6j$
  Ocean current (4 parts to east, 3 parts to south) $\longrightarrow B: 4 - 3j$
  Sum, $C: 6 + 3j \longrightarrow$ 6 parts to east, 3 parts to north



Using Complex Numbers by Substitution

- Substitution method is mathematically awkward
  - There is no equation, there is representation

Example

- When A equals B, we know countless consequences: $5A = 5B$, $1 + A = 1 + B$, $A/x = B/x$, etc.
- When A represents B, without additional information, we know nothing
- E.g., when sinusoids are represented by complex numbers, we allow addition and subtraction, but prohibit multiplication and division

Using Complex Numbers by Mathematical Equivalence

- Mathematical equivalence is a way of making complex numbers mathematically equivalent to physical problem

Example

- In DSP, sine and cosine waves can be described as having a positive frequency or a negative frequency
- Substitution method ignores negative frequencies
- Since there are applications where negative frequencies are important, mathematical equivalence is of help here

- This will be discussed throughout this course

References

STEVEN W. SMITH, *The Scientist & Engineer's Guide to Digital Signal Processing*, CALIFORNIA TECHNICAL PUB, 1997.