Digital Signal Processing

Introduction

Moslem Amiri, Václav Přenosil

Embedded Systems Laboratory
Faculty of Informatics, Masaryk University
Brno, Czech Republic

amiri@mail.muni.cz
prenosil@fi.muni.cz

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The Concept of Frequency

- Frequency is closely related to a specific type of periodic motion called harmonic oscillation
  - Described by sinusoidal functions
- Frequency has dimension of inverse time
  - Nature of time (continuous or discrete) would affect nature of frequency accordingly

Continuous-Time Sinusoidal Signals

- A simple harmonic oscillation

$$x_a(t) = A\cos(\Omega t + \theta),\ -\infty < t < \infty$$

  - Subscript a = analog signal
  - A = amplitude
  - Ω = frequency (in rad/s)
  - θ = phase (in radians)

- Rewriting above equation using frequency F in cycles per second or hertz (Hz)

$$x_a(t) = A\cos(2\pi Ft + \theta),\ -\infty < t < \infty$$

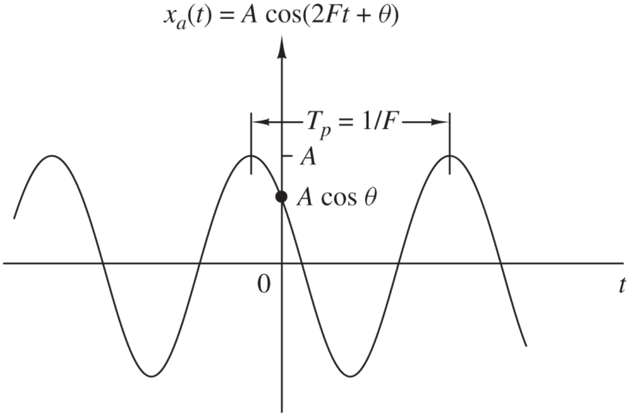


Figure 1.3.1 Example of an analog sinusoidal signal.

Continuous-Time Sinusoidal Signals

- Properties of $x_a(t) = A\cos(2\pi Ft + \theta), \ -\infty < t < \infty$
  1. For every fixed F, $x_a(t)$ is periodic
     $$x_a(t + T_p) = x_a(t)$$
     $T_p = 1/F =$ fundamental period of sinusoidal signal
  2. Signals with distinct frequencies are themselves distinct
  3. Increasing F results in an increase in rate of oscillation of signal
- Using Euler identity
  $$e^{\pm j\phi} = \cos\phi \pm j\sin\phi$$
  and introducing negative frequencies
  $$x_a(t) = A\cos(\Omega t + \theta) = \frac{A}{2}e^{j(\Omega t+\theta)} + \frac{A}{2}e^{-j(\Omega t+\theta)}$$
  - A sinusoidal signal can be obtained by adding two equal-amplitude complex-conjugate exponential signals, called **phasors**
  - As time progresses, phasors rotate in opposite directions with angular frequencies $\pm\Omega$ radians/second

Discrete-Time Sinusoidal Signals

- A discrete-time sinusoidal signal

$$x(n) = A\cos(\omega n + \theta),\ -\infty < n < \infty$$

  - $n =$ an integer called sample number
  - $A =$ amplitude
  - $\omega =$ frequency in radians/sample
  - $\theta =$ phase in radians

- Using $\omega = 2\pi f$

$$x(n) = A\cos(2\pi f n + \theta),\ -\infty < n < \infty$$

  frequency f is in cycles/sample

Discrete-Time Sinusoidal Signals

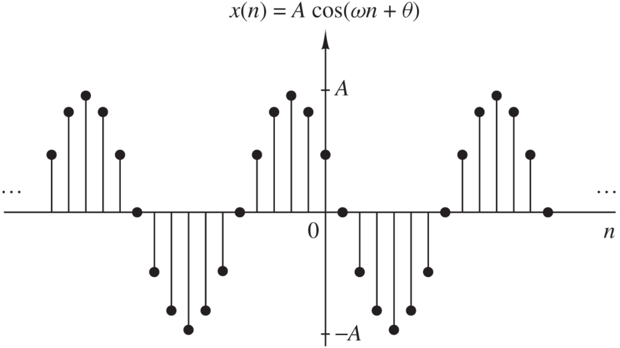


Figure 1.3.3 Example of a discrete-time sinusoidal signal ($\omega = \pi/6$ and $\theta = \pi/3$).

Discrete-Time Sinusoidal Signals

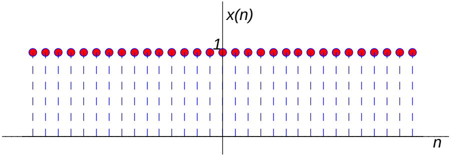
- Properties of discrete-time sinusoids
  1. A discrete-time sinusoid is periodic only if its frequency f is a rational number
     - $x(n)$ is periodic with period $N (N > 0)$ if and only if
       $$x(n+N) = x(n) \quad \text{for all } n$$
       Smallest value of N for which this equation is true is called **fundamental period**
     - Proof of this property
       $$\cos[2\pi f_0(N+n)+\theta] = \cos(2\pi f_0 n + \theta)$$
       $$2\pi f_0 N = 2k\pi$$
       $$f_0 = k/N$$
     - To determine fundamental period N, express its frequency as $f_0 = k/N$ and cancel common factors so that k and N are relatively prime, then N is answer

- Properties of discrete-time sinusoids (continued)
  2. Discrete-time sinusoids whose frequencies are separated by an integer multiple of 2π are identical

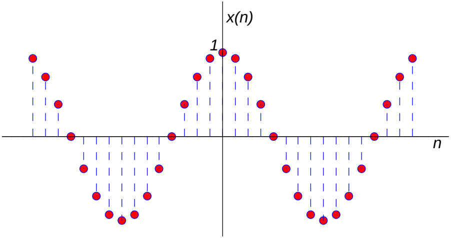
     $$\cos[(\omega_0 + 2k\pi)n + \theta] = \cos(\omega_0 n + 2\pi kn + \theta) = \cos(\omega_0 n + \theta)$$

     where $-\pi \leq \omega_0 \leq \pi$
     - Discrete-time sinusoids with $|\omega| \leq \pi$ or $|f| \leq \frac{1}{2}$ are unique
     - Any sequence resulting from a sinusoid with $|\omega| > \pi$ or $|f| > \frac{1}{2}$ is identical to a sequence obtained from a sinusoid with $|\omega| < \pi$
     - Sinusoid having $|\omega| > \pi$ is called an **alias** of a corresponding sinusoid with $|\omega| < \pi$

- Properties of discrete-time sinusoids (continued)
  3. The highest rate of oscillation in a discrete-time sinusoid is attained when $\omega = \pi$ (or $\omega = -\pi$) or, equivalently, $f = \frac{1}{2}$ (or $f = -\frac{1}{2}$)
     - $x(n) = \cos \omega_0 n$, $\omega_0 = 0 \Longrightarrow N = \infty$

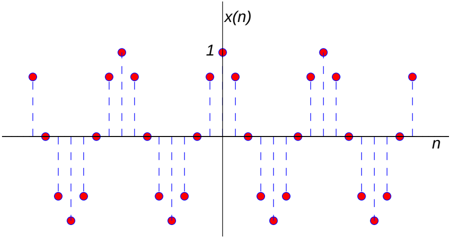


     - $x(n) = \cos \omega_0 n$, $\omega_0 = \frac{\pi}{8} \Longrightarrow N = 16$

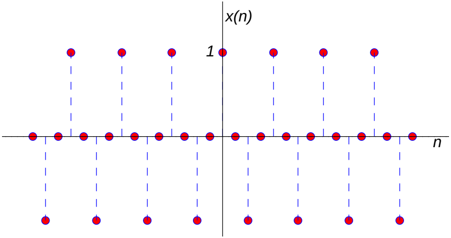


Discrete-Time Sinusoidal Signals

- Properties of discrete-time sinusoids (continued)
  3. The highest rate of oscillation is when $\omega = \pi$
     - $x(n) = \cos \omega_0 n$, $\omega_0 = \frac{\pi}{4} \implies N = 8$

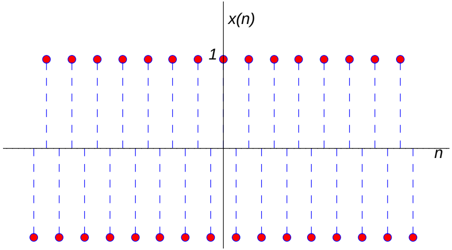


     - $x(n) = \cos \omega_0 n$, $\omega_0 = \frac{\pi}{2} \implies N = 4$



Discrete-Time Sinusoidal Signals

- Properties of discrete-time sinusoids (continued)
  3. The highest rate of oscillation is when $\omega = \pi$
     - $x(n) = \cos \omega_0 n$, $\omega_0 = \pi \implies N = 2$



- For $\pi \leq \omega_0 \leq 2\pi$, if consider sinusoids with $\omega_1 = \omega_0$ and $\omega_2 = 2\pi - \omega_0$

$$\begin{aligned} x_1(n) &= A \cos \omega_1 n = A \cos \omega_0 n \\ x_2(n) &= A \cos \omega_2 n = A \cos(2\pi - \omega_0) n \\ &= A \cos(-\omega_0 n) = x_1(n) \end{aligned}$$

  Hence, ω_2 is an alias of ω_1
- Using a sine function, result would be same, except phase difference would be π between $x_1(n)$ and $x_2(n)$

Discrete-Time Sinusoidal Signals

- Negative frequencies for discrete-time signals

$$x(n) = A\cos(\omega n + \theta) = \frac{A}{2}e^{j(\omega n+\theta)} + \frac{A}{2}e^{-j(\omega n+\theta)}$$

- Since discrete-time sinusoids with frequencies separated by $2k\pi$ are identical
  - Frequency range for discrete-time sinusoids is finite with duration 2π
  - Usually $0 \leq \omega \leq 2\pi$ or $-\pi \leq \omega \leq \pi$ is called **fundamental range**

Harmonically Related Complex Exponentials

- Harmonically related complex exponentials
  - Sets of periodic complex exponentials with fundamental frequencies that are multiples of a single positive frequency
- Properties which hold for complex exponentials, also hold for sinusoidal signals
  - We confine our discussion to complex exponentials

Continuous-Time Exponentials

- Basic signals for continuous-time, harmonically related exponentials

$$s_k(t) = e^{jk\Omega_0 t} = e^{j2\pi kF_0 t} \quad k = 0, \pm 1, \pm 2, \ldots$$

  - For each k, $s_k(t)$ is periodic with fundamental period $1/(kF_0) = T_p/k$ or fundamental frequency kF_0
  - A signal that is periodic with period T_p/k is also periodic with period $k(T_p/k) = T_p$ for any positive integer k
    - Hence all $s_k(t)$ have a common period of T_p

- A linear combination of harmonically related complex exponentials

$$x_a(t) = \sum_{k=-\infty}^{\infty} c_k s_k(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\Omega_0 t}$$

  - This is **Fourier series** expansion for $x_a(t)$
  - $c_k, k = 0, \pm 1, \pm 2, \ldots$ are arbitrary complex constants (Fourier series coefficients)
  - $s_k(t)$ is kth harmonic of $x_a(t)$
  - $x_a(t)$ is periodic with fundamental period $T_p = 1/F_0$

Discrete-Time Exponentials

- A discrete-time complex exponential is periodic if its frequency is a rational number
  - Hence, we choose $f_0 = 1/N$
- Sets of harmonically related complex exponentials

$$s_k(n) = e^{j2\pi k f_0 n}, \quad k = 0, \pm 1, \pm 2, \ldots$$

  - Since

$$s_{k+N}(n) = e^{j2\pi n(k+N)/N} = e^{j2\pi n} s_k(n) = s_k(n)$$

    there are only N distinct periodic complex exponentials in the set
  - All members of the set have a common period of N samples
  - We can choose any consecutive N complex exponentials to form a set
  - For convenience

$$s_k(n) = e^{j2\pi kn/N}, \quad k = 0, 1, 2, \ldots, N-1$$

- Fourier series representation for a periodic discrete-time sequence

$$x(n) = \sum_{k=0}^{N-1} c_k s_k(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$$

  - Fundamental period $= N$
  - Fourier coefficients $= \{c_k\}$
  - Sequence $s_k(n)$ is called kth harmonic of $x(n)$

Discrete-Time Exponentials

Example

- Stored in memory is one cycle of sinusoidal signal
$$x(n) = \sin\left(\frac{2\pi n}{N} + \theta\right)$$
where $\theta = 2\pi q/N$, where q and N are integers
- Obtain values of harmonically related sinusoids having the same phase
$$x_k(n) = \sin\left(\frac{2\pi nk}{N} + \theta\right) = \sin\left(\frac{2\pi(kn)}{N} + \theta\right) = x(kn)$$
Thus $x_k(0) = x(0), x_k(1) = x(k), x_k(2) = x(2k), \ldots$
- Obtain sinusoids of the same frequency but different phase
We control phase θ of sinusoid with $f_k = k/N$ by taking first value of sequence from memory location $q = \theta N/2\pi$, where q is an integer
We wrap around table each time index (kn) exceeds N

Analog-to-Digital and Digital-to-Analog Conversion

- Analog-to-digital (A/D) conversion
  - Converting analog signals to a sequence of numbers having finite precision
  - Corresponding devices are called A/D converters (ADCs)
- A/D conversion is a three-step process

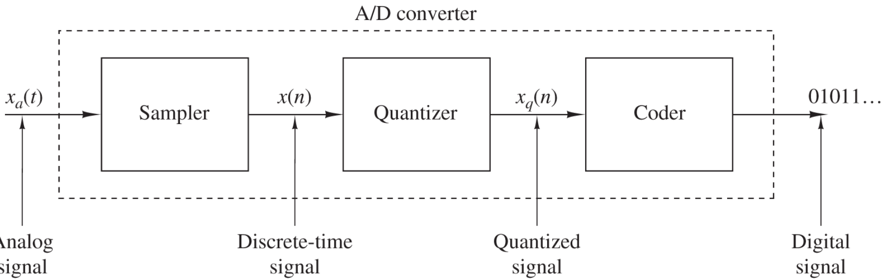


Figure 1.4.1 Basic parts of an analog-to-digital (A/D) converter.

Analog-to-Digital and Digital-to-Analog Conversion

- A/D conversion process
  1. Sampling
     - Taking samples of continuous-time signal at discrete-time instants
     - $x_a(t)$ is input $\longrightarrow x_a(nT) \equiv x(n)$ is output
     - $T =$ sampling interval
  2. Quantization
     - Conversion of a discrete-time continuous-valued signal into a discrete-time, discrete-valued signal
     - Value of each sample is selected from a finite set of possible values
     - Quantization error: Difference between unquantized sample $x(n)$ and quantized output $x_q(n)$
  3. Coding
     - Each discrete value $x_q(n)$ is represented by a b-bit binary sequence

- Digital-to-analog (D/A) conversion
  - Process of converting a digital signal into an analog signal
  - Interpolation
    - Connecting dots in a digital signal
    - Approximations: zero-order hold (staircase), linear, quadratic, and so on

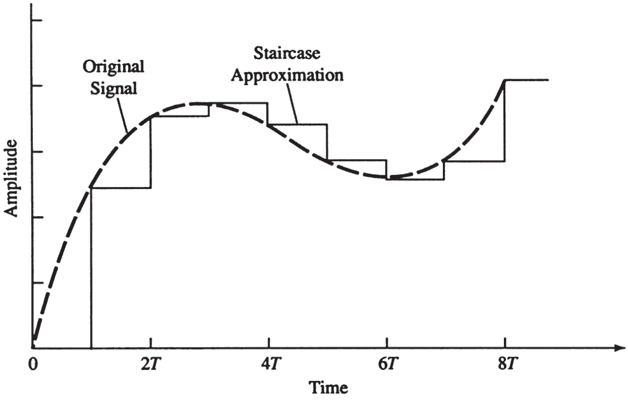


Figure 1.4.2 Zero-order hold digital-to-analog (D/A) conversion.

- Periodic or uniform sampling
  - $x(n) = x_a(nT), \quad -\infty < n < \infty$
  - T = sampling period or sample interval
  - $1/T = F_s$ = sampling rate (samples/second) or sampling frequency (hertz)

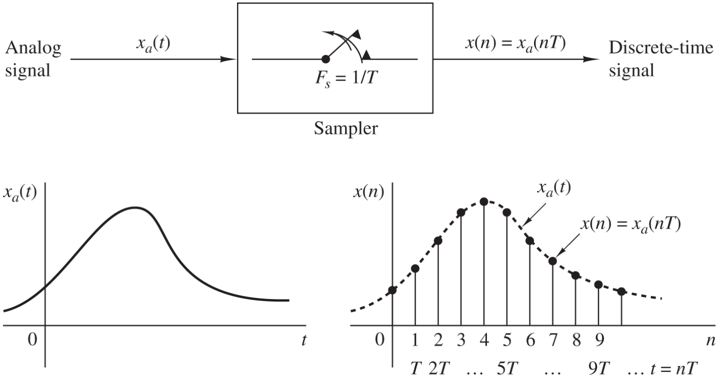


Figure 1.4.3 Periodic sampling of an analog signal.

Sampling of Analog Signals

- Relationship between t of continuous-time and n of discrete-time signals

$$t = nT = \frac{n}{F_s}$$

- To establish a relationship between F (or Ω) and f (or ω)

$$x_a(t) = A\cos(2\pi Ft + \theta)$$
$$x_a(nT) \equiv x(n) = A\cos(2\pi FnT + \theta) = A\cos\left(\frac{2\pi nF}{F_s} + \theta\right)$$
$$f = F/F_s$$
$$\omega = \Omega T$$

- Substituting $f = F/F_s$ and $\omega = \Omega T$ into following range

$$-\frac{1}{2} < f < \frac{1}{2}$$
$$-\pi < \omega < \pi$$

we find that F and Ω must fall in the range

$$-\frac{1}{2T} = -\frac{F_s}{2} \leq F \leq \frac{F_s}{2} = \frac{1}{2T}$$

$$-\frac{\pi}{T} = -\pi F_s \leq \Omega \leq \pi F_s = \frac{\pi}{T}$$

Sampling of Analog Signals

- Summary of relations among frequency variables

Continuous-time signals		Discrete-time signals
$\Omega = 2\pi F$		$\omega = 2\pi f$
$\frac{radians}{sec}$ $\quad Hz$		$\frac{radians}{sample}$ $\quad \frac{cycles}{sample}$
	$\xrightarrow{\omega=\Omega T, f=F/F_s}$	
		$-\pi \leq \omega \leq \pi$ $-\frac{1}{2} \leq f \leq \frac{1}{2}$
	$\xleftarrow{\Omega=\omega/T, F=f.F_s}$	
$-\infty < \Omega < \infty$		$-\pi/T \leq \Omega \leq \pi/T$
$-\infty < F < \infty$		$-F_s/2 \leq F \leq F_s/2$

- Since the highest frequency in a discrete-time signal is $\omega = \pi$ or $f = \frac{1}{2}$

$$F_{max} = \frac{F_s}{2} = \frac{1}{2T}$$
$$\Omega_{max} = \pi F_s = \frac{\pi}{T}$$

Example

- Consider these analog sinusoids sampled at $F_s = 40$ Hz

$$x_1(t) = \cos 2\pi(10)t$$
$$x_2(t) = \cos 2\pi(50)t$$

- Corresponding discrete-time signals

$$x_1(n) = \cos 2\pi\left(\frac{10}{40}\right)n = \cos \frac{\pi}{2}n$$
$$x_2(n) = \cos 2\pi\left(\frac{50}{40}\right)n = \cos \frac{5\pi}{2}n$$

- However

$$\cos 5\pi n/2 = \cos(2\pi n + \pi n/2) = \cos \pi n/2$$
$$x_2(n) = x_1(n)$$

- Given sampled values generated by $\cos(\pi/2)n$, there is ambiguity as to whether they correspond to $x_1(t)$ or $x_2(t)$
  - $F_2 = 50$ Hz is an **alias** of $F_1 = 10$ Hz at $F_s = 40$
  - All $\cos 2\pi(F_1 + 40k)t, k = 1, 2, \ldots$ sampled at $F_s = 40$ are aliases of $F_1 = 10$

Sampling of Analog Signals

- If sinusoids

$$x_a(t) = A\cos(2\pi F_k t + \theta)$$

where

$$F_k = F_0 + kF_s, \quad k = \pm 1, \pm 2, \ldots$$

are sampled at F_s, then F_k is outside the range $-F_s/2 \leq F \leq F_s/2$

$$\begin{aligned} x(n) \equiv x_a(nT) &= A\cos\left(2\pi \frac{F_0 + kF_s}{F_s} n + \theta\right) \\ &= A\cos(2\pi n F_0/F_s + \theta + 2\pi kn) \\ &= A\cos(2\pi f_0 n + \theta) \end{aligned}$$

- Frequencies

$$F_k = F_0 + kF_s, \quad -\infty < k < \infty$$

are **aliases** of F_0 after sampling

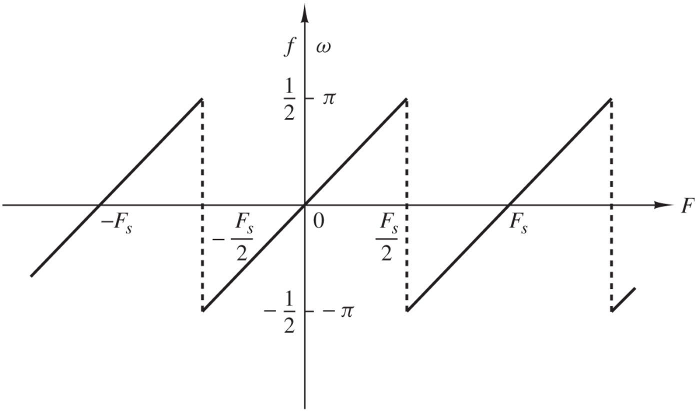


Figure 1.4.4 Relationship between the continuous-time and discrete-time frequency variables in the case of periodic sampling.

Example

- Two sinusoids: $F_1 = -\frac{7}{8}$ Hz and $F_2 = \frac{1}{8}$ Hz with $F_s = 1$ Hz

$$F_1 = F_2 + kF_s, \quad k = \pm 1, \pm 2, \ldots$$

$$k = -1, F_2 = F_1 + F_s = (-\tfrac{7}{8} + 1)Hz = \tfrac{1}{8}Hz$$

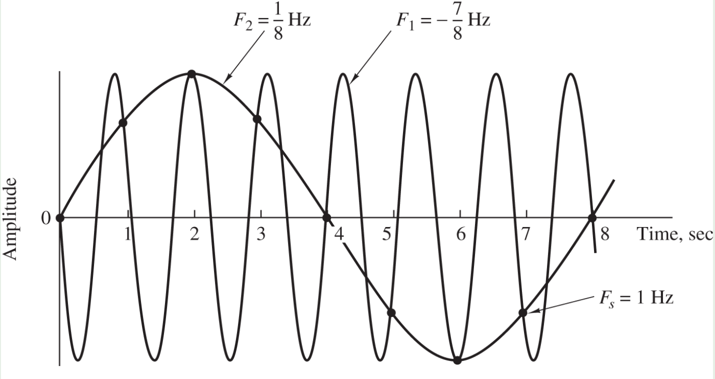


Figure 1.4.5 Illustration of aliasing.

Sampling of Analog Signals

- $F_s/2$ (which corresponds to $\omega = \pi$) is highest frequency that can be represented uniquely with F_s
  - Use $F_s/2$ or $\omega = \pi$ as pivotal point and **fold** alias frequency to range $0 \leq \omega \leq \pi$
  - $F_s/2$ ($\omega = \pi$) is called **folding frequency**

Sampling of Analog Signals

Example

$$x_a(t) = 3\cos 100\pi t$$

1. Minimum sampling rate required to avoid aliasing
   $F = 50$ Hz $\longrightarrow F_s = 100$ Hz
2. Suppose $F_s = 200$ Hz. Discrete-time signal obtained after sampling
   $x(n) = 3\cos\frac{100\pi}{200}n = 3\cos\frac{\pi}{2}n$
3. Suppose $F_s = 75$ Hz. Discrete-time signal obtained after sampling
   $x(n) = 3\cos\frac{100\pi}{75}n = 3\cos\frac{4\pi}{3}n = 3\cos\left(2\pi - \frac{2\pi}{3}\right)n = 3\cos\frac{2\pi}{3}n$
4. Frequency $0 < F < F_s/2$ of a sinusoid that yields samples identical to those obtained in part (3)
   For $F_s = 75$ Hz, $F = fF_s = 75f$
   In part (3), $f = \frac{1}{3} \longrightarrow F = 25$ Hz
   $y_a(t) = 3\cos 2\pi Ft = 3\cos 50\pi t$
   Hence $F = 50$ Hz is an alias of $F = 25$ Hz for $F_s = 75$ Hz

The Sampling Theorem

- Given any analog signal, how T or equivalently F_s should be selected
  - Knowing F_{max} of general class of signals (e.g., class of speech signals), we can specify F_s
- Suppose any analog signal can be represented as

$$x_a(t) = \sum_{i=1}^{N} A_i \cos(2\pi F_i t + \theta_i)$$

N = number of frequency components
  - F_{max} may vary slightly from different realizations among signals of any given class (e.g., from speaker to speaker)
    - To ensure F_{max} does not exceed some predetermined value, pass analog signal through a filter that attenuates frequency components above F_{max}
  - Any frequency outside $-F_s/2 \leq F \leq F_s/2$ results in samples identical with a corresponding frequency inside this range
  - To avoid aliasing

$$F_s > 2F_{max}$$

    - This condition ensures that any frequency component ($|F_i| < F_{max}$) in analog signal is mapped into a discrete-time sinusoid with a frequency $-\frac{1}{2} \leq f_i = \frac{F_i}{F_s} \leq \frac{1}{2}$

- Sampling theorem
  - If $F_{max} = B$ for $x_a(t)$ and $F_s > 2F_{max} \equiv 2B$, then $x_a(t)$ can be exactly recovered from its sample values using the interpolation function

$$g(t) = \frac{\sin 2\pi Bt}{2\pi Bt}$$

$$x_a(t) = \sum_{n=-\infty}^{\infty} x_a\left(\frac{n}{F_s}\right) g\left(t - \frac{n}{F_s}\right)$$

    where $x_a(n/F_s) = x_a(nT) \equiv x(n)$ are samples of $x_a(t)$
  - **Nyquist rate** $= F_N = 2B = 2F_{max}$

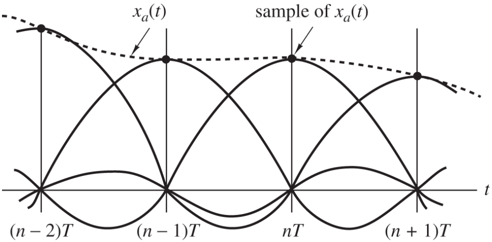


Figure 1.4.6 Ideal D/A conversion (interpolation).

Example

$$x_a(t) = 3\cos 50\pi t + 10\sin 300\pi t + \cos 100\pi t$$

- Nyquist rate for this signal

$$F_1 = 25\ Hz, \quad F_2 = 150\ Hz, \quad F_3 = 50\ Hz \longrightarrow F_{max} = 150\ Hz$$
$$F_s > 2F_{max} = 300\ Hz$$
$$F_N = 2F_{max} = 300\ Hz$$

- Component $10\sin 300\pi t$ sampled at $F_N = 300$ results in $10\sin \pi n$ which are identically zero

  If $\theta \neq 0$ or π, samples taken at Nyquist rate are not all zero

$$10\sin(\pi n + \theta) = 10(\sin \pi n \cos\theta + \cos\pi n \sin\theta) = 10\sin\theta\cos\pi n = (-1)^n 10\sin\theta$$

  To avoid this uncertain situation, sample analog signal at a rate higher than Nyquist

Example

$$x_a(t) = 3\cos 2000\pi t + 5\sin 6000\pi t + 10\cos 12000\pi t$$

- Nyquist rate for this signal

$$F_1 = 1\ kHz, \quad F_2 = 3\ kHz, \quad F_3 = 6\ kHz \longrightarrow F_{max} = 6\ kHz$$
$$F_s > 2F_{max} = 12\ kHz$$
$$F_N = 12\ kHz$$

- Assume $F_s = 5000$ samples/s. Signal obtained after sampling

$$\text{Folding frequency} = \frac{F_s}{2} = 2.5\ kHz$$
$$x(n) = x_a(nT) = x_a\left(\frac{n}{F_s}\right) = 3\cos 2\pi(\tfrac{1}{5})n + 5\sin 2\pi(\tfrac{3}{5})n +$$
$$10\cos 2\pi(\tfrac{6}{5})n = 3\cos 2\pi(\tfrac{1}{5})n + 5\sin 2\pi(1-\tfrac{2}{5})n + 10\cos 2\pi(1+\tfrac{1}{5})n =$$
$$3\cos 2\pi(\tfrac{1}{5})n + 5\sin 2\pi(-\tfrac{2}{5})n + 10\cos 2\pi(\tfrac{1}{5})n =$$
$$13\cos 2\pi(\tfrac{1}{5})n - 5\sin 2\pi(\tfrac{2}{5})n$$

Example (continued)

- Second solution:
  Aliases of F_0: $F_k = F_0 + kF_s \longrightarrow F_0 = F_k - kF_s$ such that $-F_s/2 \leq F_0 \leq F_s/2$

$$F_1 \text{ is less than } F_s/2$$
$$F_2' = F_2 - F_s = -2\ kHz$$
$$F_3' = F_3 - F_s = 1\ kHz$$
$$f = \frac{F}{F_s} \longrightarrow f_1 = \frac{1}{5}, f_2 = -\frac{2}{5}, f_3 = \frac{1}{5}$$

- Analog signal $y_a(t)$ reconstructed from samples using ideal interpolation
  Only frequency components at 1 kHz and 2 kHz are present in sampled signal

$$y_a(t) = 13\cos 2000\pi t - 5\sin 4000\pi t$$

  This distortion was caused by aliasing effect due to low F_s used

Quantization of Continuous-Amplitude Signals

- Quantization
  - Process of converting a discrete-time continuous-amplitude signal into a digital signal

$$x_q(n) = Q[x(n)] = \text{sequence of quantized samples}$$

  - Each sample value is expressed as a finite number of digits
  - Error introduced is called **quantization error** or **quantization noise**

$$e_q(n) = x_q(n) - x(n)$$

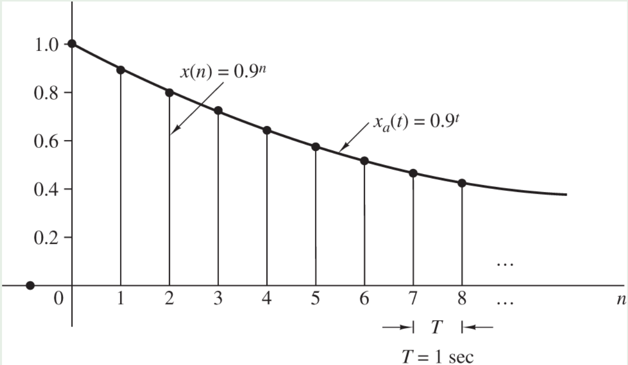
Example

- Consider discrete-time signal

$$x(n) = \begin{cases} 0.9^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

obtained by sampling $x_a(t) = 0.9^t,\ t \geq 0$ with $F_s = 1$ Hz

Example (continued)



- Following table shows values of first 10 samples of $x(n)$
  - Description of sample value $x(n)$ requires n significant digits

Quantization of Continuous-Amplitude Signals

Example (continued)

n	$x(n)$ Discrete-time signal	$x_q(n)$ (Truncation)	$x_q(n)$ (Rounding)	$e_q(n) = x_q(n) - x(n)$ (Rounding)
0	1	1.0	1.0	0.0
1	0.9	0.9	0.9	0.0
2	0.81	0.8	0.8	-0.01
3	0.729	0.7	0.7	-0.029
4	0.6561	0.6	0.7	0.0439
5	0.59049	0.5	0.6	0.00951
6	0.531441	0.5	0.5	-0.031441
7	0.4782969	0.4	0.5	0.0217031
8	0.43046721	0.4	0.4	-0.03046721
9	0.387420489	0.3	0.4	0.012579511

- Assume using one significant digit. To eliminate excess digits
  - do **truncation**
  - or do **rounding**
- Rounding process is illustrated in next figure

Example (continued)

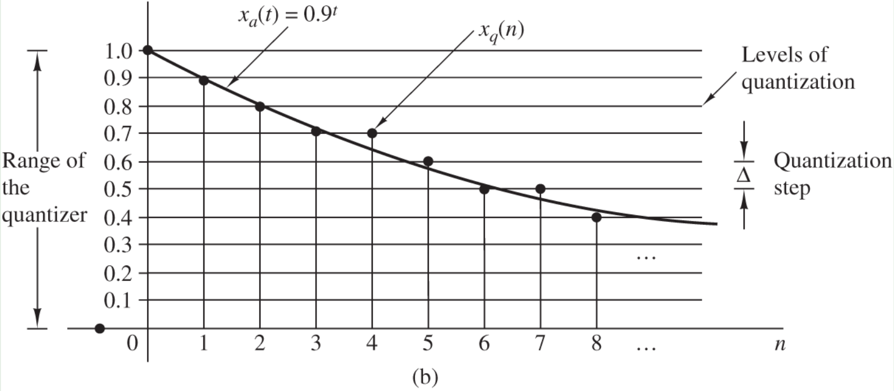


Figure 1.4.7 Illustration of quantization.

- Quantization step size (or resolution) $= \Delta = \frac{1-0}{11-1} = 0.1$

Quantization of Continuous-Amplitude Signals

- Range of quantization error $e_q(n)$ in rounding

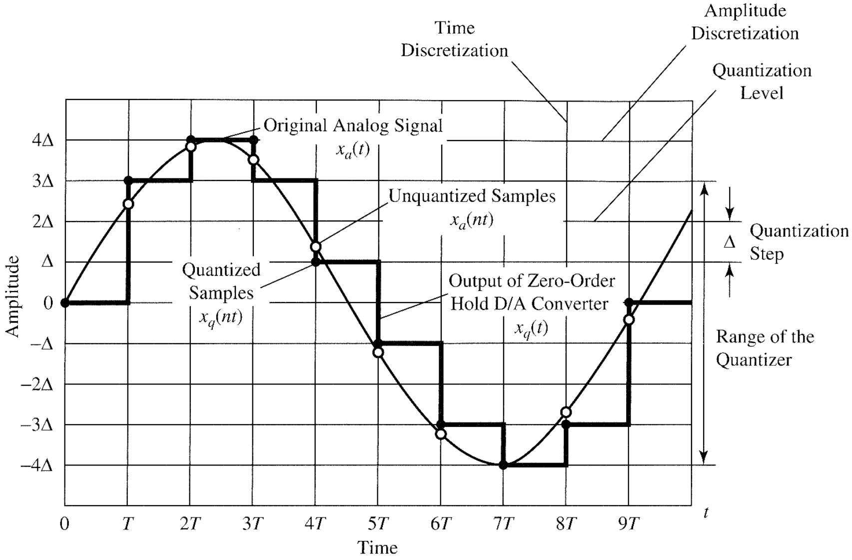
$$-\frac{\Delta}{2} \leq e_q(n) \leq \frac{\Delta}{2}$$

$$\Delta = \frac{x_{max} - x_{min}}{L - 1}$$

  - x_{min} and x_{max} = minimum and maximum value of $x(n)$
  - L = number of quantization levels
  - Dynamic range of signal = $x_{max} - x_{min}$
- Quantization of analog signals always results in a loss of information

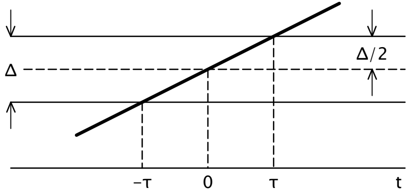
Quantization of Sinusoidal Signals

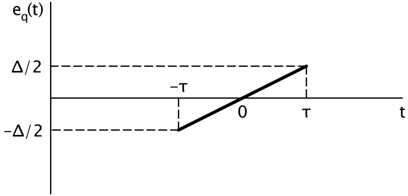
- Sampling and quantization of $x_a(t) = A\cos\Omega_0 t$



Quantization of Sinusoidal Signals

- If F_s satisfies sampling theorem, quantization is the only error in A/D process
  - Thus we can evaluate quantization error by quantizing $x_a(t)$ instead of $x(n) = x_a(nT)$
- $x_a(t)$ is almost linear between quantization levels
  - Quantization error $= e_q(t) = x_a(t) - x_q(t)$





  - τ denotes time that $x_a(t)$ stays within quantization levels

Quantization of Sinusoidal Signals

- Mean-square error power

$$P_q = \frac{1}{2\tau}\int_{-\tau}^{\tau} e_q^2(t)\,dt = \frac{1}{\tau}\int_{0}^{\tau} e_q^2(t)\,dt$$

since $e_q(t) = (\Delta/2\tau)t,\ -\tau \le t \le \tau$

$$P_q = \frac{1}{\tau}\int_{0}^{\tau}\left(\frac{\Delta}{2\tau}\right)^2 t^2\,dt = \frac{\Delta^2}{12}$$

- If quantizer has b bits of accuracy and covers range $2A$

$$\Delta = \frac{2A}{2^b} \longrightarrow P_q = \frac{A^2/3}{2^{2b}}$$

- Average power of $x_a(t)$

$$P_x = \frac{1}{T_p}\int_{0}^{T_p}(A\cos\Omega_0 t)^2\,dt = \frac{A^2}{2}$$

Quantization of Sinusoidal Signals

- Quality of output of A/D converter is measured by signal-to-quantization noise ratio **(SQNR)**
  - Provides ratio of signal power to noise power

$$SQNR = \frac{P_x}{P_q} = \frac{3}{2}.2^{2b}$$

$$SQNR(dB) = 10\log_{10} SQNR = 1.76 + 6.02b$$

Coding of Quantized Samples

- Coding process assigns a unique binary number to each quantization level
  - L levels need at least L different binary numbers

    b bits $\longrightarrow 2^b$ different binary numbers $\longrightarrow 2^b \geq L \longrightarrow b \geq \log_2 L$

References

John G. Proakis, Dimitris G. Manolakis, *Digital Signal Processing: Principles, Algorithms, and Applications*, Prentice Hall, 2006.