Digital Signal Processing

Discrete-Time Signals and Systems (2)

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Techniques for the Analysis of Linear Systems

- Methods for analyzing behavior or response of a linear system to a given input
  - First method: through difference equations (will not be discussed)
  - Second method:
    1. Decompose input signal into a weighted sum of elementary signals

$$x(n) = \sum_k c_k x_k(n)$$

    2. Using linearity property of system, responses of system to elementary signals are added to obtain total response of system
       Assuming system is relaxed

$$y_k(n) \equiv \tau[x_k(n)]$$

$$y(n) = \tau[x(n)] = \tau\left[\sum_k c_k x_k(n)\right] = \sum_k c_k \tau[x_k(n)] = \sum_k c_k y_k(n)$$

- Resolution of input signals into a weighted sum of unit sample (impulse) sequences is mathematically convenient and general

Resolution of a Discrete-Time Signal into Impulses

- An arbitrary signal $x(n)$ is to be resolved into a sum of unit sample sequences
  - We select elementary signals $x_k(n)$ to be
    $$x_k(n) = \delta(n-k)$$
  - If $x(n)$ and $\delta(n-k)$ are multiplied, result is another sequence that is zero everywhere except at $n = k$, where it is $x(k)$
    $$x(n)\delta(n-k) = x(k)\delta(n-k)$$
  - Consequently
    $$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

Resolution of a Discrete-Time Signal into Impulses

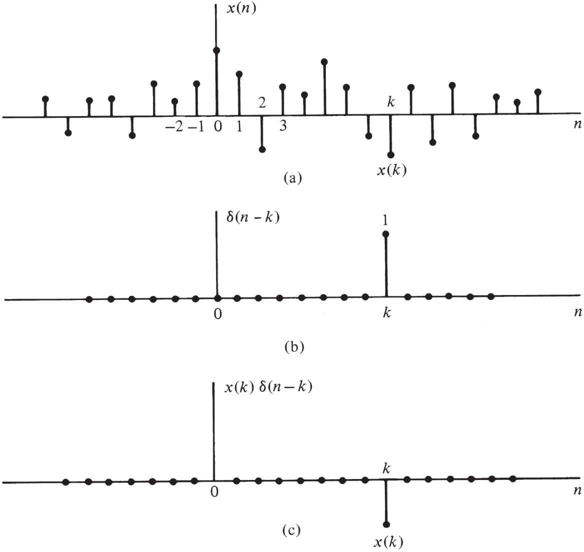

Figure 1: Multiplication of a signal $x(n)$ with a shifted unit sample sequence.

Resolution of a Discrete-Time Signal into Impulses

Example

- Resolve following finite-duration sequence into a sum of weighted impulse sequences

$$x(n) = \{2, \underset{\uparrow}{4}, 0, 3\}$$

- $x(n)$ is nonzero for $n = -1, 0, 2$

$$x(n) = 2\delta(n+1) + 4\delta(n) + 3\delta(n-2)$$

The Convolution Sum

- The response $y(n, k)$ of any relaxed linear system to the input unit sample sequence at $n = k$ is denoted by $h(n, k)$
$$y(n, k) \equiv h(n, k) = \tau[\delta(n - k)]$$
- If impulse at input is scaled by c_k, response of system is
$$c_k h(n, k) = x(k)h(n, k)$$
- For input $x(n)$
$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n - k)$$
response of system is following **superposition summation**
$$y(n) = \tau[x(n)] = \tau\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n - k)\right] = \sum_{k=-\infty}^{\infty} x(k)\tau[\delta(n - k)]$$
$$= \sum_{k=-\infty}^{\infty} x(k)h(n, k)$$

The Convolution Sum

- If response of LTI (Linear Time-Invariant) system to $\delta(n)$ is denoted as

  $$h(n) \equiv \tau[\delta(n)]$$

  then

  $$h(n-k) = \tau[\delta(n-k)]$$

  Consequently, response of system is

  $$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{1}$$

  - This formula is called a **convolution sum**
  - Input $x(n)$ is convolved with impulse response $h(n)$ to yield output $y(n)$

The Convolution Sum

- Suppose we wish to compute output of system at $n = n_0$

$$y(n_0) = \sum_{k=-\infty}^{\infty} x(k)h(n_0 - k)$$

Process of computing convolution between $x(k)$ and $h(k)$:

1. **Folding**. Fold $h(k)$ about $k = 0$ to obtain $h(-k)$
2. **Shifting**. Shift $h(-k)$ by n_0 to right (left) if n_0 is positive (negative) to obtain $h(n_0 - k)$
3. **Multiplication**. $\upsilon_{n_0}(k) \equiv x(k)h(n_0 - k)$
4. **Summation**. Sum all values of $\upsilon_{n_0}(k)$ to obtain output at $n = n_0$

The Convolution Sum

Example

- Impulse response of an LTI system is

$$h(n) = \{1, \underset{\uparrow}{2}, 1, -1\}$$

Determine response of system to input signal

$$x(n) = \{\underset{\uparrow}{1}, 2, 3, 1\}$$

- To compute output at $n = 0$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \longrightarrow y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k)$$

  - First fold $h(k)$ - no shifting is required - then do multiplication

$$v_0(k) \equiv x(k)h(-k)$$

  - Finally, sum of all terms in product sequence yields

$$y(0) = \sum_{k=-\infty}^{\infty} v_0(k) = 4$$

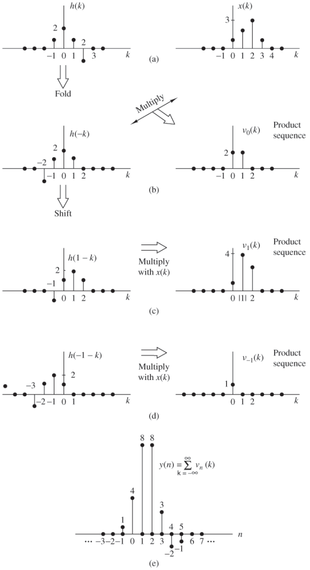

Figure 2: Graphical computation of convolution.

Example (continued)

- Response of system at $n = 1$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \longrightarrow y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k)$$

  - $h(1-k)$ is $h(-k)$ shifted to right by one unit
  - Product sequence

$$\upsilon_1(k) = x(k)h(1-k)$$

  - Sum of all values in product sequence

$$y(1) = \sum_{k=-\infty}^{\infty} \upsilon_1(k) = 8$$

- By shifting $h(-k)$ farther to right, multiplying and summing, we obtain

$$y(2) = 8,\ y(3) = 3,\ y(4) = -2,\ y(5) = -1$$

- For $n > 5$, $y(n) = 0$ because product sequences contain all zeros

The Convolution Sum

Example (continued)

- To evaluate $y(n)$ for $n = -1$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \longrightarrow y(-1) = \sum_{k=-\infty}^{\infty} x(k)h(-1-k)$$

  - $h(-1-k)$ is $h(-k)$ shifted one unit to left
  - Product sequence
    $$v_{-1}(k) = x(k)h(-1-k)$$
  - Sum of all values in product sequence
    $$y(-1) = \sum_{k=-\infty}^{\infty} v_{-1}(k) = 1$$
- Further shifts of $h(-1-k)$ to left result in all-zero product sequence
  $$y(n) = 0 \quad \text{for } n \leq -2$$
- Entire response of system for $-\infty < n < \infty$
  $$y(n) = \{\ldots, 0, 0, 1, \underset{\uparrow}{4}, 8, 8, 3, -2, -1, 0, 0, \ldots\}$$

The Convolution Sum

- Convolution operation is commutative
  - It is irrelevant which of two sequences is folded and shifted

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \xrightarrow{m=n-k} y(n) = \sum_{m=-\infty}^{\infty} x(n-m)h(m)$$

$$\xrightarrow{\text{replace } m \text{ by } k} y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \qquad (2)$$

- Product sequences in (1) and (2) are not identical
  - If

$$\upsilon_n(k) = x(k)h(n-k)$$
$$\omega_n(k) = x(n-k)h(k)$$

then

$$\upsilon_n(k) = \omega_n(n-k)$$

therefore

$$y(n) = \sum_{k=-\infty}^{\infty} \upsilon_n(k) = \sum_{k=-\infty}^{\infty} \omega_n(n-k)$$

  - Both sequences contain same values in a different arrangement

The Convolution Sum

Example

- Determine output $y(n)$ of a relaxed LTI system with impulse response
$$h(n) = a^n u(n), \ |a| < 1$$
when input is a unit step sequence: $x(n) = u(n)$
- We use
$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k)$$

The Convolution Sum

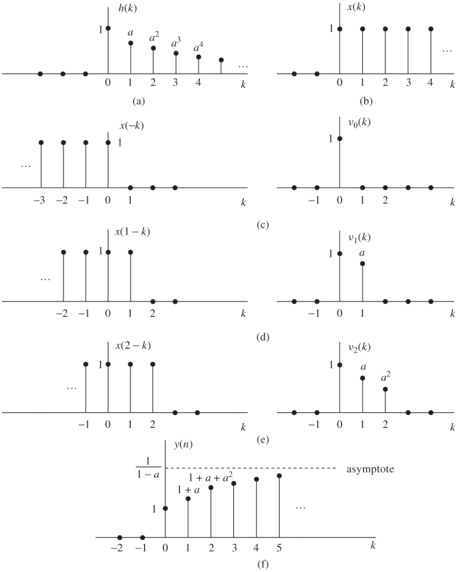

Figure 3: Graphical computation of convolution example.

Example (continued)

- We obtain outputs

$$y(0) = 1$$
$$y(1) = 1 + a$$
$$y(2) = 1 + a + a^2$$

  for $n > 0$

$$y(n) = 1 + a + a^2 + \ldots + a^n = \frac{1 - a^{n+1}}{1 - a}$$

- For $n < 0$, product sequences consist of all zeros. Hence

$$y(n) = 0, \quad n < 0$$

- Since $|a| < 1$

$$y(\infty) = \lim_{n \to \infty} y(n) = \frac{1}{1 - a}$$

Properties of Convolution - Interconnection of LTI Systems

- An **asterisk** is used to denote convolution operation

$$y(n) = x(n) * h(n) \equiv \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

$$y(n) = h(n) * x(n) \equiv \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

- **Identity** and **shifting** properties
  - $\delta(n)$ is identity element for convolution
  $$y(n) = x(n) * \delta(n) = x(n)$$
  - Shifting $\delta(n)$ by k, convolution sequence is also shifted by k
  $$x(n) * \delta(n-k) = y(n-k) = x(n-k)$$

Properties of Convolution - Interconnection of LTI Systems

- **Commutative law**

$$x(n) * h(n) = h(n) * x(n)$$

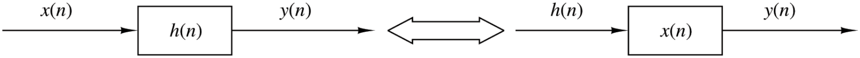

Figure 4: Interpretation of the commutative property of convolution.

- **Associative law**

$$[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$$

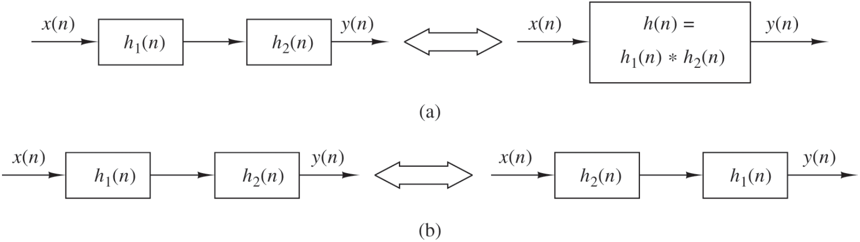

Figure 5: Implications of the associative (a) and the associative and commutative (b) properties of convolution.

Example

- Determine impulse response for cascade of two LTI systems having impulse responses

$$h_1(n) = (\tfrac{1}{2})^n u(n) \quad \text{and} \quad h_2(n) = (\tfrac{1}{4})^n u(n)$$

- Convolve $h_1(n)$ and $h_2(n)$

$$h(n) = \textstyle\sum_{k=-\infty}^{\infty} h_1(k) h_2(n-k)$$

$$\upsilon_n(k) = h_1(k) h_2(n-k) = (\frac{1}{2})^k u(k) (\frac{1}{4})^{n-k} u(n-k)$$

  - $\upsilon_n(k)$ is nonzero for $k \geq 0$ and $n - k \geq 0$ (or $n \geq k \geq 0$)

$$h(n) = \sum_{k=0}^{n} (\frac{1}{2})^k (\frac{1}{4})^{n-k} = (\frac{1}{4})^n \sum_{k=0}^{n} 2^k$$

$$= (\frac{1}{4})^n (2^{n+1} - 1) = (\frac{1}{2})^n [2 - (\frac{1}{2})^n], \; n \geq 0$$

  - For $n < 0 \longrightarrow \upsilon_n(k) = 0$ for all $k \longrightarrow h(n) = 0, \; n < 0$

- **Distributive law**

$$x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$$

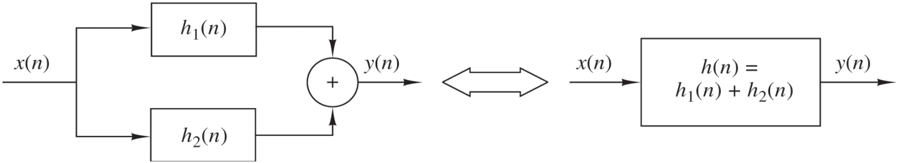

Figure 6: Interpretation of the distributive property of convolution: two LTI systems connected in parallel can be replaced by a single system with $h(n) = h_1(n) + h_2(n)$.

Causal Linear Time-Invariant Systems

- For an LTI system, causality can be translated to a condition on impulse response
- Consider an LTI system at time $n = n_0$

$$y(n_0) = \sum_{k=-\infty}^{\infty} h(k)x(n_0 - k) = \sum_{k=0}^{\infty} h(k)x(n_0 - k) + \sum_{k=-\infty}^{-1} h(k)x(n_0 - k)$$

  - First sum: present and past inputs ($x(n)$ for $n \leq n_0$)
  - Second sum: future inputs ($x(n)$ for $n > n_0$)
- If output at $n = n_0$ is to depend only on present and past inputs, then

$$h(n) = 0, \quad n < 0$$

- *An LTI system is causal iff its $h(n) = 0$ for negative values of n.* Thus

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n - k) = \sum_{k=-\infty}^{n} x(k)h(n - k)$$

Causal Linear Time-Invariant Systems

- A sequence that is zero for $n < 0$ is called a **causal sequence**
  - If nonzero for $n < 0$ and $n > 0$, it is called a **noncausal sequence**
- If input to a causal LTI system is a causal sequence

$$y(n) = \sum_{k=0}^{n} h(k)x(n-k) = \sum_{k=0}^{n} x(k)h(n-k)$$

Example

- Determine unit step response of LTI system with impulse response

$$h(n) = a^n u(n), \quad |a| < 1$$

- Both input signal (unit step) and system are causal

$$y(n) = \sum_{k=0}^{n} h(k)x(n-k) = \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}$$

  - $y(n) = 0$ for $n < 0$

- Taking absolute value of both sides of convolution formula, we obtain

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} h(k)x(n-k) \right| \leq \sum_{k=-\infty}^{\infty} |h(k)||x(n-k)|$$

  - If input is bounded, there exists a finite number M_x such that $|x(n)| \leq M_x$

$$|y(n)| \leq M_x \sum_{k=-\infty}^{\infty} |h(k)|$$

  - Output is bounded if

$$S_h \equiv \sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

  - *An LTI system is stable if its impulse response is absolutely summable*
  - This condition implies that $h(n)$ goes to zero as n approaches infinity

- Suppose $|x(n)| < M_x$ for $n < n_0$ and $x(n) = 0$ for $n \geq n_0$

$$y(n_0 + N) = \sum_{k=-\infty}^{N-1} h(k)x(n_0 + N - k) + \sum_{k=N}^{\infty} h(k)x(n_0 + N - k)$$

  - First sum is zero since $x(n) = 0$ for $n \geq n_0$

$$|y(n_0 + N)| = \left| \sum_{k=N}^{\infty} h(k)x(n_0 + N - k) \right| \leq \sum_{k=N}^{\infty} |h(k)||x(n_0 + N - k)|$$

$$\leq M_x \sum_{k=N}^{\infty} |h(k)|$$

$$\lim_{N\to\infty} \sum_{k=N}^{\infty} |h(k)| = 0 \longrightarrow \lim_{N\to\infty} |y(n_0 + N)| = 0$$

  - In a stable system, any finite duration input produces a transient output

Stability of Linear Time-Invariant Systems

Example

- Determine range of values of parameter a for which LTI system with $h(n) = a^n u(n)$ is stable
- System is causal

$$S_h \equiv \sum_{k=-\infty}^{\infty} |h(k)| \longrightarrow \sum_{k=0}^{\infty} |a^k| = \sum_{k=0}^{\infty} |a|^k = 1 + |a| + |a|^2 + \cdots$$

Geometric series converges to

$$\sum_{k=0}^{\infty} |a|^k = \frac{1}{1-|a|}$$

provided that $|a| < 1$. Therefore, system is stable if $|a| < 1$
Otherwise, it diverges and becomes unstable

Example

- Determine range of a and b for which following LTI system is stable

$$h(n) = \begin{cases} a^n, & n \geq 0 \\ b^n, & n < 0 \end{cases}$$

- System is noncausal

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=0}^{\infty} |a|^n + \sum_{n=-\infty}^{-1} |b|^n$$

$$\sum_{n=-\infty}^{-1} |b|^n = \sum_{n=1}^{\infty} \frac{1}{|b|^n} = \frac{1}{|b|}\left(1 + \frac{1}{|b|} + \frac{1}{|b|^2} + \cdots\right) = \frac{1/|b|}{1 - 1/|b|}$$

where $1/|b| < 1$

System is stable if $|a| < 1$ and $|b| > 1$

- We can subdivide LTI systems into two types
  1. Those having a finite-duration impulse response (FIR)
  2. Those having an infinite-duration impulse response (IIR)
- For causal FIR systems

$$h(n) = 0, \quad n < 0 \text{ and } n \geq M$$
$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$

  - FIR system acts as a *window* that views only most recent M input samples in forming output
  - Thus, FIR system has a finite memory of length-M samples
- For causal IIR systems

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$$

  - IIR system has an infinite memory

Correlation of Discrete-Time Signals

- Correlation closely resembles convolution
  - But objective in computing correlation between two signals is to measure the degree to which they are similar

Crosscorrelation and Autocorrelation Sequences

- For two real signal sequences $x(n)$ and $y(n)$ each having finite energy
  - **Crosscorrelation** of $x(n)$ and $y(n)$ is a sequence $r_{xy}(l)$

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)y(n-l) = \sum_{n=-\infty}^{\infty} x(n+l)y(n), \qquad l = 0, \pm 1, \pm 2, \ldots$$

  - Index l is (time) shift (or *lag*) parameter
  - Reversing roles of $x(n)$ and $y(n)$

$$r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n)x(n-l) = \sum_{n=-\infty}^{\infty} y(n+l)x(n), \qquad l = 0, \pm 1, \pm 2, \ldots$$

$$r_{xy}(l) = r_{yx}(-l)$$

  - $r_{yx}(l)$ is folded version of $r_{xy}(l)$, where folding is about $l = 0$
  - Hence, $r_{yx}(l)$ provides exactly same info as $r_{xy}(l)$, with respect to similarity of $x(n)$ to $y(n)$

Crosscorrelation and Autocorrelation Sequences

Example

- Determine crosscorrelation sequence $r_{xy}(l)$ of sequences

$$x(n) = \{\ldots, 0, 0, 2, -1, 3, 7, \underset{\uparrow}{1}, 2, -3, 0, 0, \ldots\}$$

$$y(n) = \{\ldots, 0, 0, 1, -1, 2, -2, \underset{\uparrow}{4}, 1, -2, 5, 0, 0, \ldots\}$$

- For $l = 0$

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)y(n-l) \xrightarrow{l=0} r_{xy}(0) = \sum_{n=-\infty}^{\infty} x(n)y(n)$$

$$\upsilon_0(n) = x(n)y(n) = \{\ldots, 0, 2, 1, 6, -14, \underset{\uparrow}{4}, 2, 6, 0, \ldots\} \longrightarrow r_{xy}(0) = 7$$

- For $l > 0$ ($l < 0$), shift $y(n)$ to right (left) relative to $x(n)$ by l units, compute $\upsilon_l(n) = x(n)y(n-l)$, and sum over all values of $\upsilon_l(n)$

$$r_{xy}(l) = \{10, -9, 19, 36, -14, 33, 0, \underset{\uparrow}{7}, 13, -18, 16, -7, 5, -3\}$$

- Except for folding operation in convolution, computations of crosscorrelation and convolution are similar
$$r_{xy}(l) = x(l) * y(-l)$$
- In special case where $y(n) = x(n)$, we have **autocorrelation** of $x(n)$
$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)x(n-l) = \sum_{n=-\infty}^{\infty} x(n+l)x(n)$$
- If $x(n)$ and $y(n)$ are causal sequences of length N
$$r_{xy}(l) = \sum_{n=i}^{N-|k|-1} x(n)y(n-l)$$
$$r_{xx}(l) = \sum_{n=i}^{N-|k|-1} x(n)x(n-l)$$
where $i = l$, $k = 0$ for $l \geq 0$, and $i = 0$, $k = l$ for $l < 0$

Properties of Autocorrelation & Crosscorrelation Sequences

- Assume $x(n)$ and $y(n)$ with finite energy and their linear combination
$$ax(n) + by(n-l)$$
Energy in this signal
$$\sum_{n=-\infty}^{\infty} [ax(n) + by(n-l)]^2 = a^2 \sum_{n=-\infty}^{\infty} x^2(n) + b^2 \sum_{n=-\infty}^{\infty} y^2(n-l) + 2ab \sum_{n=-\infty}^{\infty} x(n)y(n-l)$$
$$= a^2 r_{xx}(0) + b^2 r_{yy}(0) + 2ab r_{xy}(l)$$

  - $r_{xx}(0) = E_x =$ energy of $x(n)$
  - $r_{yy}(0) = E_y =$ energy of $y(n)$

Properties of Autocorrelation & Crosscorrelation Sequences

- It is obvious
$$a^2 r_{xx}(0) + b^2 r_{yy}(0) + 2ab r_{xy}(l) \geq 0$$
Assuming $b \neq 0$
$$r_{xx}(0)\left(\frac{a}{b}\right)^2 + 2r_{xy}(l)\left(\frac{a}{b}\right) + r_{yy}(0) \geq 0$$
Since this quadratic is nonnegative, its discriminant is nonpositive
$$4[r_{xy}^2(l) - r_{xx}(0)r_{yy}(0)] \leq 0$$
$$|r_{xy}(l)| \leq \sqrt{r_{xx}(0)r_{yy}(0)} = \sqrt{E_x E_y}$$
When $y(n) = x(n)$
$$|r_{xx}(l)| \leq r_{xx}(0) = E_x$$
This means max value of autocorrelation of a signal is at zero lag
- By scaling signals, shape of crosscorrelation sequence does not change
    - Only amplitudes of crosscorrelation sequence are scaled accordingly
    - Since scaling is unimportant, auto and crosscorrelation sequences are normalized to range from -1 to 1, in practice

$$\rho_{xx}(l) = \frac{r_{xx}(l)}{r_{xx}(0)} \quad \text{and} \quad \rho_{xy}(l) = \frac{r_{xy}(l)}{\sqrt{r_{xx}(0)r_{yy}(0)}}$$

Properties of Autocorrelation & Crosscorrelation Sequences

- As shown before

$$r_{xy}(l) = r_{yx}(-l)$$

With $y(n) = x(n)$

$$r_{xx}(l) = r_{xx}(-l)$$

  - Hence autocorrelation is an even function
  - It suffices to compute $r_{xx}(l)$ for $l \geq 0$

Example

- Compute autocorrelation of $x(n) = a^n u(n), \quad 0 < a < 1$
- If $l \geq 0$

$$r_{xx}(l) = \sum_{n=l}^{\infty} x(n)x(n-l) = \sum_{n=l}^{\infty} a^n a^{n-l} = a^{-l} \sum_{n=l}^{\infty} (a^2)^n = \frac{1}{1-a^2} a^l$$

If $l < 0$

$$r_{xx}(l) = \sum_{n=0}^{\infty} x(n)x(n-l) = a^{-l} \sum_{n=0}^{\infty} (a^2)^n = \frac{1}{1-a^2} a^{-l}$$

$$r_{xx}(l) = \frac{1}{1-a^2} a^{|l|}, \quad -\infty < l < \infty$$

$$r_{xx}(0) = \frac{1}{1-a^2} \xrightarrow[\text{autocorrelation}]{\text{normalized}} \rho_{xx}(l) = \frac{r_{xx}(l)}{r_{xx}(0)} = a^{|l|}, \quad -\infty < l < \infty$$

Properties of Autocorrelation & Crosscorrelation Sequences

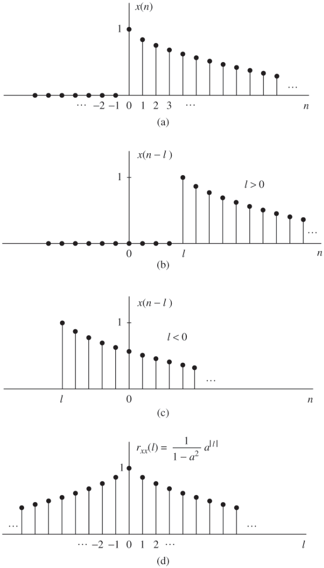

Figure 7: Computation of the autocorrelation of the signal $x(n) = a^n, 0 < a < 1$.

Correlation of Periodic Sequences

- If $x(n)$ and $y(n)$ are power signals

$$r_{xy}(l) = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} x(n)y(n-l)$$

$$r_{xx}(l) = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} x(n)x(n-l)$$

- If $x(n)$ and $y(n)$ are two periodic sequences, each with period N

$$r_{xy}(l) = \frac{1}{N} \sum_{n=0}^{N-1} x(n)y(n-l)$$

$$r_{xx}(l) = \frac{1}{N} \sum_{n=0}^{N-1} x(n)x(n-l)$$

Correlation of Periodic Sequences

- Correlation can be used to identify periodicities in an observed physical signal which may be corrupted by random interference

$$y(n) = x(n) + \omega(n)$$

  - $x(n)$ is a periodic sequence of unknown period N
  - $\omega(n)$ is an additive random interference
  - Suppose we observe M samples of $y(n)$

$$0 \leq n \leq M-1,\ M >> N,\ y(n) = 0 \text{ for } n < 0 \text{ and } n \geq M$$

$$r_{yy}(l) = \frac{1}{M}\sum_{n=0}^{M-1} y(n)y(n-l) = \frac{1}{M}\sum_{n=0}^{M-1}[x(n)+\omega(n)][x(n-l)+\omega(n-l)]$$

$$= \frac{1}{M}\sum_{n=0}^{M-1} x(n)x(n-l) + \frac{1}{M}\sum_{n=0}^{M-1}[x(n)\omega(n-l)+\omega(n)x(n-l)]$$

$$+ \frac{1}{M}\sum_{n=0}^{M-1} \omega(n)\omega(n-l) = r_{xx}(l) + r_{x\omega}(l) + r_{\omega x}(l) + r_{\omega\omega}(l)$$

Correlation of Periodic Sequences

- $r_{xx}(l)$ will contain large peaks at $l = 0, N, 2N$, and so on
- $r_{x\omega}(l)$ and $r_{\omega x}(l)$ will be small since $x(n)$ and $\omega(n)$ are unrelated
- $r_{\omega\omega}(l)$ will contain a peak at $l = 0$, but because of its random characteristics will decay rapidly toward zero
- Consequently, only $r_{xx}(l)$ will have large peaks for $l > 0$, so we can detect presence of periodic signal $x(n)$ and identify its period

Input-Output Correlation Sequences

- $x(n)$ with known $r_{xx}(l)$ is applied to an LTI system with $h(n)$ producing
$$y(n) = h(n) * x(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$
- Crosscorrelation between output and input signal
$$\begin{aligned} r_{yx}(l) &= y(l) * x(-l) = h(l) * [x(l) * x(-l)] \\ &= h(l) * r_{xx}(l) \end{aligned}$$
Replacing l by $-l$
$$r_{xy}(l) = h(-l) * r_{xx}(l)$$
- Autocorrelation of output signal
$$r_{yy}(l) = y(l) * y(-l) = [h(l) * x(l)] * [h(-l) * x(-l)] = [h(l) * h(-l)] * [x(l) * x(-l)] = r_{hh}(l) * r_{xx}(l)$$
  - $r_{hh}(l)$ exists if system is stable. Stability insures that system does not change type (energy or power) of input signal
  - $l = 0$ provides energy (or power) of output in terms of autocorrelations
$$r_{yy}(0) = \sum_{k=-\infty}^{\infty} r_{hh}(k)r_{xx}(k)$$

References

John G. Proakis, Dimitris G. Manolakis, *Digital Signal Processing: Principles, Algorithms, and Applications*, Prentice Hall, 2006.