Digital Signal Processing

Frequency Analysis of Signals (1)

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Frequency Analysis of Continuous-Time Signals

- Frequency analysis of a signal is resolution of signal into its frequency (sinusoidal) components
  - For class of periodic signals, such a decomposition is called a **Fourier series**
  - For class of finite energy signals, the decomposition is called **Fourier transform**
- The term **spectrum** is used when referring to frequency content of a signal
  - Different signal waveforms have different spectra
  - Thus spectrum provides an identity or a signature for a signal (no other signal has the same spectrum)
- Process of obtaining spectrum of a signal using basic mathematical tools is **frequency** or **spectral analysis**
  - In contrast, process of determining spectrum of a signal in practice, based on actual measurements of signal, is called **spectrum estimation**
  - In a practical problem, the signal is some information-bearing signal which does not lend itself to an exact mathematical description
- Recombination of sinusoidal components to reconstruct original signal is a Fourier synthesis problem

The Fourier Series for Continuous-Time Periodic Signals

- Examples of periodic signals are square waves, rectangular waves, triangular waves, sinusoids and complex exponentials
- Basic mathematical representation of periodic signals is Fourier series
  - Fourier series is a linear weighted sum of harmonically related sinusoids or complex exponentials
- A linear combination of harmonically related complex exponentials of form

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t} \quad (1)$$

is a periodic signal with fundamental period $T_p = 1/F_0$

  - This is called a **Fourier series**

The Fourier Series for Continuous-Time Periodic Signals

- Given a periodic signal $x(t)$ with period T_p, it can be represented by Fourier series
  - Fundamental frequency F_0 is reciprocal of given period T_p
  - To determine expression for $\{c_k\}$, first multiply both sides of (1) by

$$e^{-j2\pi F_0 l t} \qquad (l \text{ is an integer})$$

and then integrate both sides of resulting equation from t_0 to $t_0 + T_p$

$$\int_{t_0}^{t_0+T_p} x(t) e^{-j2\pi l F_0 t} dt = \int_{t_0}^{t_0+T_p} e^{-j2\pi l F_0 t} \left(\sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t} \right) dt$$

$$= \sum_{k=-\infty}^{\infty} c_k \int_{t_0}^{t_0+T_p} e^{j2\pi F_0 (k-l) t} dt \qquad (2)$$

$$= \sum_{k=-\infty}^{\infty} c_k \left[\frac{e^{j2\pi F_0 (k-l) t}}{j2\pi F_0 (k-l)} \right]_{t_0}^{t_0+T_p} \qquad (3)$$

For $k \neq l$, (3) yields zero

- If $k = l$, from equation (2)

$$\int_{t_0}^{t_0+T_p} dt = t\Big|_{t_0}^{t_0+T_p} = T_p$$

Consequently

$$\int_{t_0}^{t_0+T_p} x(t)e^{-j2\pi lF_0 t}dt = c_l T_p$$

$$c_l = \frac{1}{T_p}\int_{t_0}^{t_0+T_p} x(t)e^{-j2\pi lF_0 t}dt = \frac{1}{T_p}\int_{T_p} x(t)e^{-j2\pi lF_0 t}dt$$

Table 1: Frequency analysis of continuous-time periodic signals

Synthesis equation	$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kF_0 t}$
Analysis equation	$c_k = \frac{1}{T_p}\int_{T_p} x(t)e^{-j2\pi kF_0 t}dt$

The Fourier Series for Continuous-Time Periodic Signals

- Fourier coefficients c_k are complex valued
  - If periodic signal is real, c_k and c_{-k} are complex conjugates

    $$c_k = |c_k|e^{j\theta_k} \longrightarrow c_{-k} = |c_k|e^{-j\theta_k}$$

    Consequently, Fourier series may also be represented as

    $$x(t) = c_0 + 2\sum_{k=1}^{\infty} |c_k| \cos(2\pi k F_0 t + \theta_k)$$

    where c_0 is real when $x(t)$ is real
  - Expanding cosine function in equation above

    $$\cos(2\pi k F_0 t + \theta_k) = \cos 2\pi k F_0 t \cos\theta_k - \sin 2\pi k F_0 t \sin\theta_k$$

    $$x(t) = a_0 + \sum_{k=1}^{\infty}(a_k \cos 2\pi k F_0 t - b_k \sin 2\pi k F_0 t)$$

    where

    $$a_0 = c_0$$
    $$a_k = 2|c_k| \cos\theta_k$$
    $$b_k = 2|c_k| \sin\theta_k$$

Power Density Spectrum of Periodic Signals

- A periodic signal has infinite energy and a finite average power, given as

$$P_x = \frac{1}{T_p} \int_{T_p} |x(t)|^2 dt$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t} \longrightarrow P_x = \frac{1}{T_p} \int_{T_p} x(t) \sum_{k=-\infty}^{\infty} c_k^* e^{-j2\pi k F_0 t} dt$$

$$= \sum_{k=-\infty}^{\infty} c_k^* \left[\frac{1}{T_p} \int_{T_p} x(t) e^{-j2\pi k F_0 t} dt \right]$$

$$= \sum_{k=-\infty}^{\infty} |c_k|^2$$

The established relation is called **Parseval's relation** for power signals

$$P_x = \frac{1}{T_p} \int_{T_p} |x(t)|^2 d(t) = \sum_{k=-\infty}^{\infty} |c_k|^2$$

Power Density Spectrum of Periodic Signals

Example

- Suppose $x(t)$ consists of a single complex exponential

$$x(t) = c_k e^{j2\pi k F_0 t}$$

In this case, all Fourier series coefficients except c_k are zero

$$P_x = \sum_{k=-\infty}^{\infty} |c_k|^2 \longrightarrow P_x = |c_k|^2$$

  - $|c_k|^2$ represents power in kth harmonic component of signal
  - Hence total average power in periodic signal is simply sum of average powers in all harmonics

Power Density Spectrum of Periodic Signals

- Plotting $|c_k|^2$ as a function of frequencies $kF_0, k = 0, \pm1, \pm2, \ldots$, obtained diagram is called **power density spectrum** or **power spectrum** of periodic signal $x(t)$

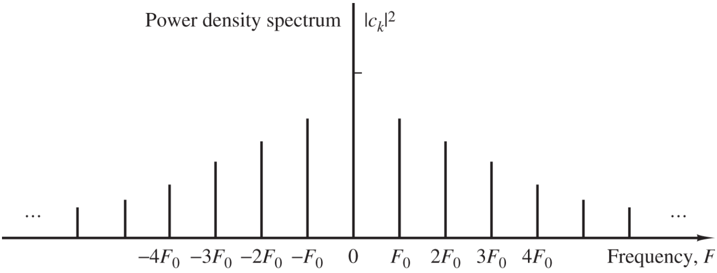

Figure 1: Power density spectrum of a continuous-time periodic signal.

- Since power in a periodic signal exists only at discrete values of frequencies, the signal is said to have a **line spectrum**

Power Density Spectrum of Periodic Signals

- Since Fourier series coefficients $\{c_k\}$ are complex valued, i.e.,
$$c_k = |c_k|e^{j\theta_k} \quad \text{where} \quad \theta_k = \measuredangle c_k$$
instead of plotting power density spectrum, we can plot magnitude spectrum $\{|c_k|\}$ and phase spectrum $\{\theta_k\}$ as a function of frequency
- If periodic signal is real valued, then
$$c_{-k} = c_k^* \longrightarrow |c_{-k}|^2 = |c_k^*|^2 = |c_k|^2$$
  - Power spectrum is an even function
  - Magnitude spectrum is an even function
  - Phase spectrum is an odd function
  - Hence it is sufficient to specify spectrum for positive frequencies only
  - Total average power
$$P_x = c_0^2 + 2\sum_{k=1}^{\infty}|c_k|^2 = a_0^2 + \frac{1}{2}\sum_{k=1}^{\infty}(a_k^2 + b_k^2)$$
$$a_0 = c_0$$
$$a_k = 2|c_k|\cos\theta_k$$
$$b_k = 2|c_k|\sin\theta_k$$

Example

- Determine Fourier series and power density spectrum of following rectangular pulse train signal

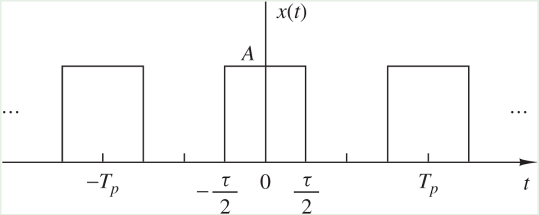

Figure 2: Continuous-time periodic train of rectangular pulses.

- Since $x(t)$ is an even signal, it is convenient to select integration interval from $-T_p/2$ to $T_p/2$

$$c_k = \frac{1}{T_p}\int_{T_p} x(t)e^{-j2\pi kF_0 t}dt \rightarrow c_0 = \frac{1}{T_p}\int_{-T_p/2}^{T_p/2} x(t)dt = \frac{1}{T_p}\int_{-\tau/2}^{\tau/2} A dt = \frac{A\tau}{T_p}$$

Example (continued)

- c_0 represents average value (dc component) of $x(t)$
- For $k \neq 0$

$$
\begin{aligned}
c_k &= \frac{1}{T_p}\int_{-\tau/2}^{\tau/2} Ae^{-j2\pi kF_0 t}dt = \frac{A}{T_p}\left[\frac{e^{-j2\pi F_0 kt}}{-j2\pi kF_0}\right]_{-\tau/2}^{\tau/2} \\
&= \frac{A}{\pi F_0 k T_p}\frac{e^{j\pi kF_0\tau} - e^{-j\pi kF_0\tau}}{j2} \\
&= \frac{A\tau}{T_p}\frac{\sin \pi kF_0\tau}{\pi kF_0\tau}, \qquad k = \pm 1, \pm 2, \ldots
\end{aligned} \tag{4}
$$

(4) has the from $(\sin\phi)/\phi$, where $\phi = \pi kF_0\tau$

- ϕ takes on discrete values since F_0 and τ are fixed and k varies
- However, plot of $(\sin\phi)/\phi$ with ϕ as a continuous parameter is shown in Fig. 3

Example (continued)

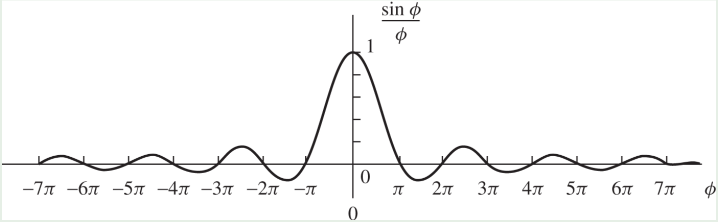

Figure 3: The function $(\sin \phi)/\phi$.

- Since $x(t)$ is even, Fourier coefficients $\{c_k\}$ are real
  - Phase spectrum is either zero, when c_k is positive, or π when c_k is negative
  - Instead of plotting magnitude and phase spectra separately, we may plot $\{c_k\}$ on a single graph

Power Density Spectrum of Periodic Signals

Example (continued)

- When T_p is fixed and pulse width τ is allowed to vary
  - $T_p = 0.25$ seconds $\longrightarrow F_0 = 1/T_p = 4$ Hz
  - Spacing between adjacent spectral lines is $F_0 = 4$ Hz, independent of τ

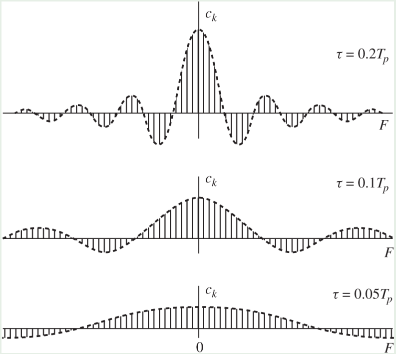

Figure 4: Fourier coefficients of the rectangular pulse train when T_p is fixed and the pulse width τ varies.

Example (continued)

- If τ is fixed and T_p varies when $T_p > \tau$
  - Spacing between adjacent spectral lines decreases as T_p increases

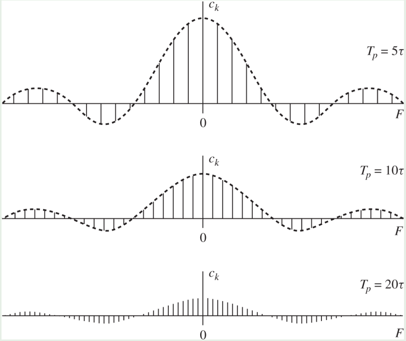

Figure 5: Fourier coefficients of a rectangular pulse train with fixed pulse width τ and varying period T_p.

Power Density Spectrum of Periodic Signals

Example (continued)

$$c_k = \frac{A\tau}{T_p} \frac{\sin \pi k F_0 \tau}{\pi k F_0 \tau}, \qquad k = \pm 1, \pm 2, \ldots$$

- If $k \neq 0$ and $\sin(\pi k F_0 \tau) = 0$, then $c_k = 0$
  - Harmonics with zero power occur at frequencies kF_0 such that
    $$\pi(kF_0)\tau = m\pi, \qquad m = \pm 1, \pm 2, \ldots$$
- Power density spectrum for rectangular pulse train

$$|c_k|^2 = \begin{cases} \left(\frac{A\tau}{T_p}\right)^2, & k = 0 \\ \left(\frac{A\tau}{T_p}\right)^2 \left(\frac{\sin \pi k F_0 \tau}{\pi k F_0 \tau}\right)^2, & k = \pm 1, \pm 2, \ldots \end{cases}$$

Fourier Transform for Continuous-Time Aperiodic Signals

- Periodic signals possess line spectra with equidistant lines
  - Line spacing is equal to fundamental frequency
  - Fundamental period provides number of lines per unit of frequency (line density), as shown in Fig. 5
- Allowing period to increase without limit, line spacing tends toward zero
  - When period becomes infinite, signal becomes aperiodic and its spectrum becomes continuous
  - Spectrum of an aperiodic signal is envelope of line spectrum in corresponding periodic signal obtained by repeating the aperiodic signal with some period T_p

- Consider an aperiodic signal $x(t)$ with finite duration
  - We can create a periodic signal $x_p(t)$ with period T_p

$$x(t) = \lim_{T_p \to \infty} x_p(t)$$

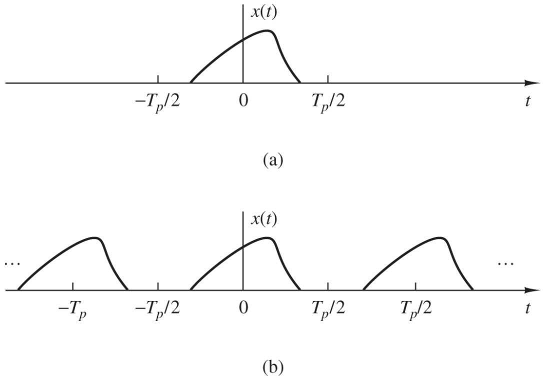

Figure 6: (a) Aperiodic signal $x(t)$ and (b) periodic signal $x_p(t)$ constructed by repeating $x(t)$ with a period T_p.

Fourier Transform for Continuous-Time Aperiodic Signals

- Fourier series representation of $x_p(t)$

$$x_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kF_0 t}, \qquad F_0 = \frac{1}{T_p}$$

where

$$c_k = \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} x_p(t) e^{-j2\pi kF_0 t} dt$$

Since $x_p(t) = x(t)$ for $-T_p/2 \leq t \leq T_p/2$ and $x(t) = 0$ for $|t| > T_p/2$

$$c_k = \frac{1}{T_p} \int_{-\infty}^{\infty} x(t) e^{-j2\pi kF_0 t} dt$$

Defining a function $X(F)$, called **Fourier transform** of $x(t)$

$$X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi Ft} dt$$

$$c_k = \frac{1}{T_p} X(kF_0) \quad \text{or} \quad T_p c_k = X(kF_0) = X\left(\frac{k}{T_p}\right)$$

Fourier coefficients are samples of $X(F)$ taken at multiples of F_0 and scaled by F_0 (multiplied by $1/T_p$)

- If we substitute for c_k in Fourier series representation of $x_p(t)$

$$x_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kF_0 t}, \qquad F_0 = \frac{1}{T_p}$$

$$T_p c_k = X(kF_0) = X\left(\frac{k}{T_p}\right)$$

we obtain

$$x_p(t) = \frac{1}{T_p} \sum_{k=-\infty}^{\infty} X\left(\frac{k}{T_p}\right) e^{j2\pi kF_0 t}$$

Defining $\Delta F = \frac{1}{T_p}$

$$x_p(t) = \sum_{k=-\infty}^{\infty} X(k\Delta F) e^{j2\pi k\Delta Ft} \Delta F$$

$$\lim_{T_p \to \infty} x_p(t) = x(t) = \lim_{\Delta F \to 0} \sum_{k=-\infty}^{\infty} X(k\Delta F) e^{j2\pi k\Delta Ft} \Delta F$$

$$\xrightarrow[k\Delta F \to F]{\Delta F \to dF} x(t) = \int_{-\infty}^{\infty} X(F) e^{j2\pi Ft} dF \quad \textbf{(inverse Fourier transform)}$$

Table 2: Frequency analysis of continuous-time aperiodic signals

Synthesis equation (inverse transform)	$x(t) = \int_{-\infty}^{\infty} X(F) e^{j2\pi Ft} dF$
Analysis equation (direct transform)	$X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi Ft} dt$

- Above Fourier transform pair can be expressed in terms of $\Omega = 2\pi F$
  - Since $dF = d\Omega / 2\pi$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} d\Omega$$

$$X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

Energy Density Spectrum of Aperiodic Signals

- Let $x(t)$ be any finite energy signal with Fourier transform $X(F)$

$$\begin{aligned} E_x &= \int_{-\infty}^{\infty} |x(t)|^2 dt \\ &= \int_{-\infty}^{\infty} x(t)x^*(t) dt \\ &= \int_{-\infty}^{\infty} x(t) dt \left[\int_{-\infty}^{\infty} X^*(F) e^{-j2\pi Ft} dF \right] \\ &= \int_{-\infty}^{\infty} X^*(F) dF \left[\int_{-\infty}^{\infty} x(t) e^{-j2\pi Ft} dt \right] \\ &= \int_{-\infty}^{\infty} |X(F)|^2 dF \end{aligned}$$

  - **Parseval's relation** for aperiodic, finite energy signals

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(F)|^2 dF$$

Energy Density Spectrum of Aperiodic Signals

- Spectrum $X(F)$ of a signal is complex valued
$$X(F) = |X(F)|e^{j\Theta(F)}$$
where $|X(F)|$ is magnitude spectrum and $\Theta(F)$ is phase spectrum
$$\Theta(F) = \measuredangle X(F)$$
- **Energy density spectrum** of $x(t)$
$$S_{xx}(F) = |X(F)|^2$$
  - $S_{xx}(F)$ is real and nonnegative, and does not contain any phase information
    - It is impossible to reconstruct signal given $S_{xx}(F)$
- If signal $x(t)$ is real, then
$$|X(-F)| = |X(F)|$$
$$\measuredangle X(-F) = -\measuredangle X(F)$$
  - Energy density spectrum of a real signal has even symmetry
$$S_{xx}(-F) = S_{xx}(F)$$

Example

- Determine Fourier transform and energy density spectrum of

$$x(t) = \begin{cases} A, & |t| \leq \tau/2 \\ 0, & |t| > \tau/2 \end{cases}$$

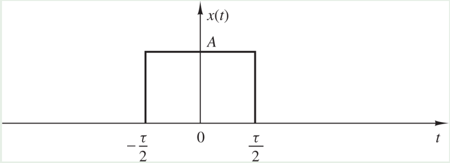

Figure 7: Rectangular pulse.

- This signal is aperiodic

$$X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi Ft} dt \rightarrow X(F) = \int_{-\tau/2}^{\tau/2} A e^{-j2\pi Ft} dt = A\tau \frac{\sin \pi F\tau}{\pi F\tau}$$

Example (continued)

- $X(F)$ is real and hence can be depicted using only one diagram

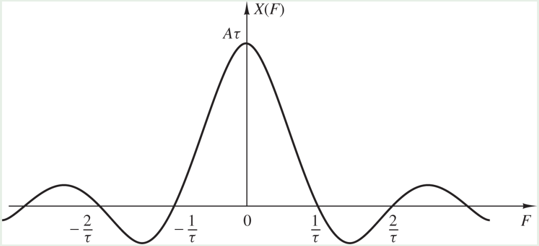

Figure 8: Fourier transform of rectangular pulse.

- Zero crossings of $X(F)$ occur at multiples of $1/\tau$
- Width of main lobe, which contains most of signal energy, is $2/\tau$
  - As τ decreases (increases), main lobe becomes broader (narrower) and more energy is moved to higher (lower) frequencies

Example (continued)

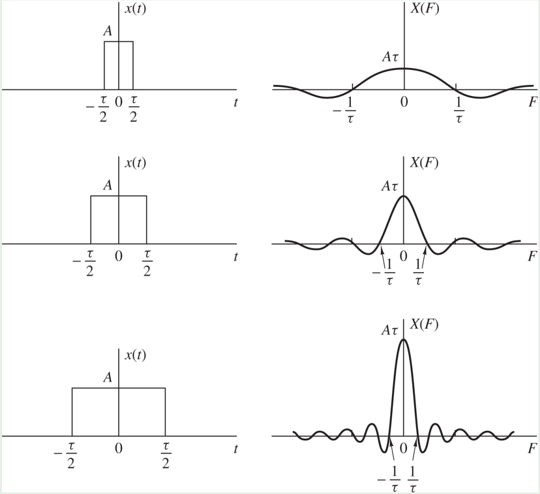

Figure 9: Fourier transform of a rectangular pulse for various width values.

Energy Density Spectrum of Aperiodic Signals

Example (continued)

- As shown in Fig. 9, as signal pulse is expanded (compressed) in time, its transform is compressed (expanded) in frequency
- Energy density spectrum of rectangular pulse

$$S_{xx}(F) = (A\tau)^2 \left(\frac{\sin \pi F \tau}{\pi F \tau} \right)^2$$

Frequency Analysis of Discrete-Time Signals

- Fourier series representation of a continuous-time periodic signal can consist of an infinite number of frequency components
  - Frequency spacing between two successive harmonically related frequencies is $1/T_p$ (T_p is fundamental period)
- Since frequency range for continuous-time signals extends from $-\infty$ to ∞, it is possible to have signals that contain an infinite number of frequency components
  - In contrast, frequency range for discrete-time signals is unique over interval $(-\pi, \pi)$ or $(0, 2\pi)$
- A discrete-time signal of fundamental period N can consist of frequency components separated by $2\pi/N$ radians or $f = 1/N$ cycles
  - Consequently, Fourier series representation of discrete-time periodic signal contains at most N frequency components

The Fourier Series for Discrete-Time Periodic Signals

- Given a periodic sequence $x(n)$ with period N (i.e., $x(n) = x(n+N)$ for all n), Fourier series representation for $x(n)$ consists of N harmonically related exponential functions

$$e^{j2\pi kn/N}, \qquad k = 0, 1, \ldots, N-1$$

and is expressed as

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N} \tag{5}$$

where $\{c_k\}$ are coefficients in series representation

- Multiplying both sides of (5) by $e^{-j2\pi ln/N}$ and summing from $n = 0$ to $n = N - 1$

$$\sum_{n=0}^{N-1} x(n)e^{-j2\pi ln/N} = \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} c_k e^{j2\pi(k-l)n/N} \tag{6}$$

$$\sum_{n=0}^{N-1} e^{j2\pi(k-l)n/N} = \begin{cases} N, & k-l = 0, \pm N, \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$$

Right-hand side of (6) reduces to Nc_l and hence

$$c_l = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi ln/N}, \qquad l = 0, 1, \dots, N-1$$

Table 3: Frequency analysis of discrete-time periodic signals

Synthesis equation	$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$
Analysis equation	$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$

- The synthesis equation above is often called **discrete-time Fourier series (DTFS)**
- From Analysis equation above, which holds for every value of k, we have

$$c_{k+N} = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi(k+N)n/N} = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} = c_k$$

  - *Spectrum of a signal* $x(n)$*, which is periodic with period* N*, is a periodic sequence with period* N
  - Any N consecutive samples of signal or its spectrum provide a complete description of signal in time or frequency domains

Example

- Determine spectrum of signal

$$x(n) = \cos \sqrt{2}\pi n$$

- Since $\omega_0 = \sqrt{2}\pi \longrightarrow f_0 = 1/\sqrt{2}$
  - f_0 is not a rational number $\longrightarrow$ signal is not periodic $\longrightarrow$ signal cannot be expanded in a Fourier series
  - Nevertheless, signal possesses a spectrum consisting of single frequency component at $\omega = \omega_0 = \sqrt{2}\pi$

Example

- Determine spectrum of signal

$$x(n) = \cos \pi n/3$$

- $f_0 = \frac{1}{6} \longrightarrow x(n)$ is periodic with fundamental period $N = 6$

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \rightarrow c_k = \frac{1}{6}\sum_{n=0}^{5} x(n)e^{-j2\pi kn/6}, k = 0, 1, \ldots, 5$$

$$x(n) = \cos\frac{\pi n}{3} = \cos\frac{2\pi n}{6} = \frac{1}{2}e^{j2\pi n/6} + \frac{1}{2}e^{-j2\pi n/6}$$

Comparing $x(n)$ with synthesis equation, $x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$

$$e^{j2\pi n/6} \rightarrow k = 1 \rightarrow c_1 = \tfrac{1}{2}$$

$$e^{-j2\pi n/6}\ (k = -1) \rightarrow e^{-j2\pi n/6} = e^{j2\pi(5-6)n/6} = e^{j2\pi(5n)/6} \rightarrow k = 5 \rightarrow c_5 = \tfrac{1}{2}$$

$$c_0 = c_2 = c_3 = c_4 = 0,\ c_1 = \tfrac{1}{2},\ c_5 = \tfrac{1}{2}$$

Example (continued)

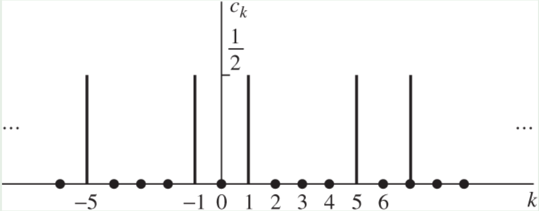

Figure 10: Spectrum of the periodic signal discussed in Example.

Example

- Determine spectrum of signal

$$x(n) = \{\underset{\uparrow}{1}, 1, 0, 0\}$$

  where $x(n)$ is periodic with period $N = 4$

- From analysis equation

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \rightarrow c_k = \frac{1}{4}\sum_{n=0}^{3} x(n)e^{-j2\pi kn/4},\ k = 0, 1, 2, 3$$

$$c_k = \tfrac{1}{4}(1 + e^{-j\pi k/2}),\ k = 0, 1, 2, 3$$
$$c_0 = \tfrac{1}{2},\ c_1 = \tfrac{1}{4}(1 - j),\ c_2 = 0,\ c_3 = \tfrac{1}{4}(1 + j)$$

  Magnitude and phase spectra are

$$|c_0| = \tfrac{1}{2},\ |c_1| = \tfrac{\sqrt{2}}{4},\ |c_2| = 0,\ |c_3| = \tfrac{\sqrt{2}}{4}$$
$$\measuredangle c_0 = 0,\ \measuredangle c_1 = -\tfrac{\pi}{4},\ \measuredangle c_2 = \text{undefined},\ \measuredangle c_3 = \tfrac{\pi}{4}$$

Example (continued)

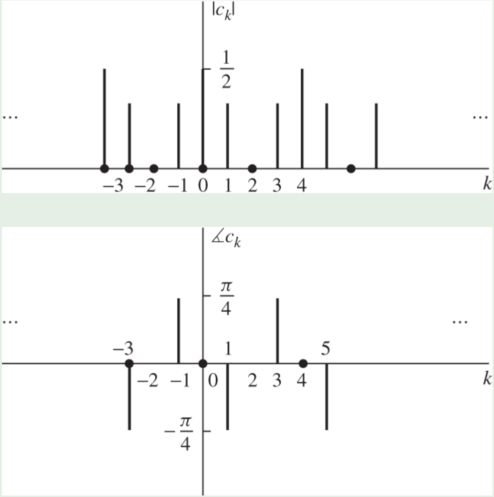

Figure 11: Spectra of the periodic signal discussed in Example.

Power Density Spectrum of Periodic Signals

- Average power of a discrete-time periodic signal with period N

$$P_x = \frac{1}{N}\sum_{n=0}^{N-1}|x(n)|^2$$

$$= \frac{1}{N}\sum_{n=0}^{N-1}x(n)x^*(n)$$

$$x(n) = \sum_{k=0}^{N-1}c_k e^{j2\pi kn/N} \rightarrow P_x = \frac{1}{N}\sum_{n=0}^{N-1}x(n)\left(\sum_{k=0}^{N-1}c_k^* e^{-j2\pi kn/N}\right)$$

$$= \sum_{k=0}^{N-1}c_k^*\left[\frac{1}{N}\sum_{n=0}^{N-1}x(n)e^{-j2\pi kn/N}\right]$$

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1}x(n)e^{-j2\pi kn/N} \rightarrow P_x = \sum_{k=0}^{N-1}|c_k|^2$$

  - Average power in signal is sum of powers of individual frequency components
  - $|c_k|^2$ for $k = 0, 1, \ldots, N-1$ is called **power density spectrum**

Power Density Spectrum of Periodic Signals

- If $x(n)$ is real ($x^*(n) = x(n)$), then

$$c_k^* = c_{-k}$$

or

$$|c_{-k}| = |c_k| \quad \text{and} \quad -\measuredangle c_{-k} = \measuredangle c_k$$

  - Because

$$c_{k+N} = c_k$$

  then

$$|c_k| = |c_{N-k}| \quad \text{and} \quad \measuredangle c_k = -\measuredangle c_{N-k}$$

  - Thus, for a real signal, the spectrum

$$c_k,\ k = 0, 1, \ldots, N/2 \text{ for } N \text{ even,}$$
$$\text{or } c_k,\ k = 0, 1, \ldots, (N-1)/2 \text{ for } N \text{ odd}$$

  completely specifies signal in frequency domain
  - This is consistent with the fact that the highest relative frequency that can be represented by a discrete-time signal is π

$$0 \leq \omega_k = 2\pi k/N \leq \pi \longrightarrow 0 \leq k \leq N/2$$

Example

- Determine Fourier series coefficients and power density spectrum of periodic signal shown in Fig. 12

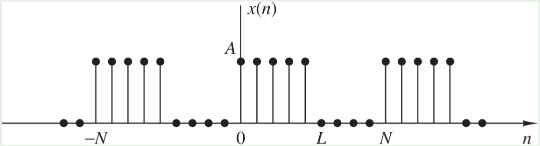

Figure 12: Discrete-time periodic square-wave signal.

- Applying analysis equation

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} = \frac{1}{N}\sum_{n=0}^{L-1} Ae^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1$$

$$c_k = \frac{A}{N}\sum_{n=0}^{L-1}(e^{-j2\pi k/N})^n = \begin{cases} \frac{AL}{N}, & k = 0 \\ \frac{A}{N}\frac{1-e^{-j2\pi kL/N}}{1-e^{-j2\pi k/N}}, & k = 1, 2, \ldots, N-1 \end{cases}$$

Example (continued)

- Simplifying last expression further

$$\frac{1-e^{-j2\pi kL/N}}{1-e^{-j2\pi k/N}} = \frac{e^{-j\pi kL/N}}{e^{-j\pi k/N}} \frac{e^{j\pi kL/N}-e^{-j\pi kL/N}}{e^{j\pi k/N}-e^{-j\pi k/N}} = e^{-j\pi k(L-1)/N} \frac{\sin(\pi kL/N)}{\sin(\pi k/N)}$$

Therefore

$$c_k = \begin{cases} \frac{AL}{N}, & k = 0, \pm N, \pm 2N, \ldots \\ \frac{A}{N} e^{-j\pi k(L-1)/N} \frac{\sin(\pi kL/N)}{\sin(\pi k/N)}, & \text{otherwise} \end{cases}$$

Power density spectrum

$$|c_k|^2 = \begin{cases} \left(\frac{AL}{N}\right)^2, & k = 0, \pm N, \pm 2N, \ldots \\ \left(\frac{A}{N}\right)^2 \left(\frac{\sin(\pi kL/N)}{\sin(\pi k/N)}\right)^2, & \text{otherwise} \end{cases}$$

Power Density Spectrum of Periodic Signals

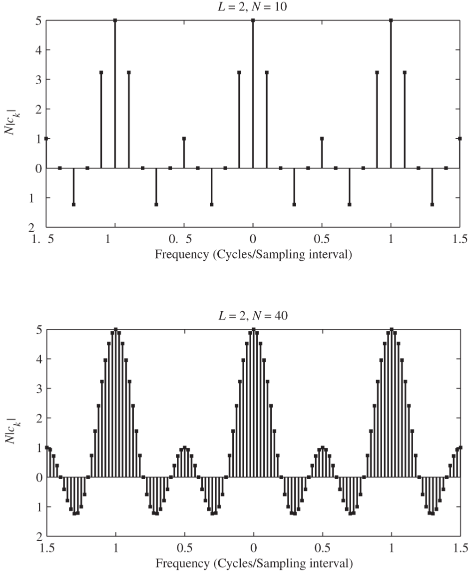

Figure 13: Power density spectrum $|c_k|^2$ for $L = 2$, $N = 10$ and 40, and $A = 1$.

The Fourier Transform of Discrete-Time Aperiodic Signals

- Fourier transform of a finite-energy discrete-time signal $x(n)$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \tag{7}$$

  - $X(\omega)$ is a decomposition of $x(n)$ into its frequency components
  - Frequency range for a discrete-time signal is unique over frequency interval of $(-\pi, \pi)$ or, equivalently, $(0, 2\pi)$

$$\begin{aligned} X(\omega + 2\pi k) &= \sum_{n=-\infty}^{\infty} x(n)e^{-j(\omega+2\pi k)n} \\ &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}e^{-j2\pi kn} \\ &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = X(\omega) \end{aligned}$$

  Hence $X(\omega)$ is periodic with period 2π

The Fourier Transform of Discrete-Time Aperiodic Signals

- Multiplying both sides of (7) by $e^{j\omega m}$ and integrating over $(-\pi, \pi)$

$$\int_{-\pi}^{\pi} X(\omega) e^{j\omega m} d\omega = \int_{-\pi}^{\pi} \left[\sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \right] e^{j\omega m} d\omega$$

On right-hand side, order interchange of $\sum$ and $\int$ can be made if

$$X_N(\omega) = \sum_{n=-N}^{N} x(n) e^{-j\omega n}$$

converges uniformly to $X(\omega)$ as $N \to \infty$

- I.e., for every ω, $X_N(\omega) \to X(\omega)$, as $N \to \infty$

$$\sum_{n=-\infty}^{\infty} x(n) \int_{-\pi}^{\pi} e^{j\omega(m-n)} d\omega = \begin{cases} 2\pi x(m), & m = n \\ 0, & m \neq n \end{cases}$$

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

The Fourier Transform of Discrete-Time Aperiodic Signals

Table 4: Frequency analysis of discrete-time aperiodic signals

Synthesis equation (inverse transform)	$x(n) = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{j\omega n} d\omega$
Analysis equation (direct transform)	$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

Energy Density Spectrum of Aperiodic Signals

- Energy of a discrete-time signal $x(n)$ is

$$\begin{aligned}
E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 &= \sum_{n=-\infty}^{\infty} x(n)x^*(n) \\
&= \sum_{n=-\infty}^{\infty} x(n) \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(\omega) e^{-j\omega n} d\omega \right] \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(\omega) \left[\sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \right] d\omega \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega
\end{aligned}$$

Parseval's relation for discrete-time aperiodic signals with finite energy

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

Energy Density Spectrum of Aperiodic Signals

- $X(\omega)$ is a complex-valued function

$$X(\omega) = |X(\omega)|e^{j\Theta(\omega)}$$

  where

$$\Theta(\omega) = \measuredangle X(\omega)$$

  is phase spectrum and $|X(\omega)|$ is magnitude spectrum
- **Energy density spectrum** of $x(n)$

$$S_{xx}(\omega) = |X(\omega)|^2$$

- if $x(n)$ is real, then

$$X^*(\omega) = X(-\omega)$$

  - $|X(-\omega)| = |X(\omega)|$
  - $\measuredangle X(-\omega) = -\measuredangle X(\omega)$
  - $S_{xx}(-\omega) = S_{xx}(\omega)$
  - Similar to real discrete-time periodic signals, frequency range of real discrete-time aperiodic signals can also be limited further to one-half of period

$$0 \leq \omega \leq \pi$$

Energy Density Spectrum of Aperiodic Signals

Example

- Determine and sketch energy density spectrum $S_{xx}(\omega)$ of signal

$$x(n) = a^n u(n), \qquad -1 < a < 1$$

- Applying Fourier transform

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n$$

Since $|ae^{-j\omega}| = |a| < 1$, using geometric series

$$X(\omega) = \frac{1}{1 - ae^{-j\omega}}$$

$$S_{xx}(\omega) = |X(\omega)|^2 = X(\omega)X^*(\omega) = \frac{1}{(1 - ae^{-j\omega})(1 - ae^{j\omega})}$$

$$= \frac{1}{1 - 2a\cos\omega + a^2}$$

Energy Density Spectrum of Aperiodic Signals

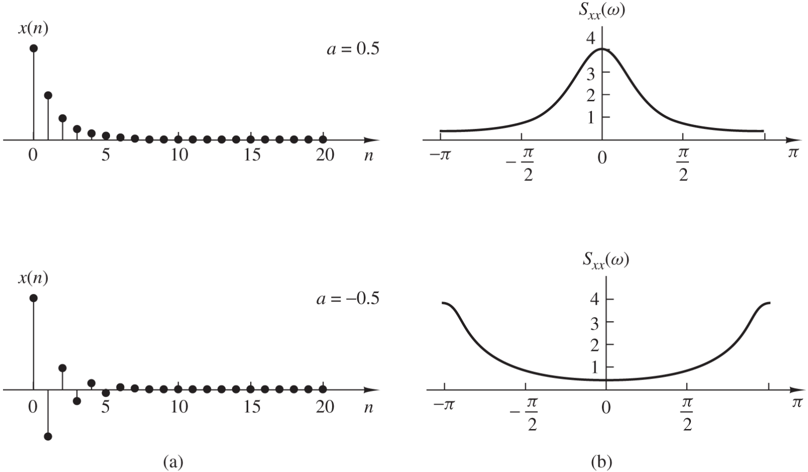

Figure 14: (a) Sequence $x(n) = (\frac{1}{2})^n u(n)$ and $x(n) = (-\frac{1}{2})^n u(n)$; (b) their energy spectra. For $a = -0.5$ the signal has more rapid variations and as a result its spectrum has stronger high frequencies.

Example

- Determine Fourier transform and energy density spectrum of sequence

$$x(n) = \begin{cases} A, & 0 \leq n \leq L-1 \\ 0, & \text{otherwise} \end{cases}$$

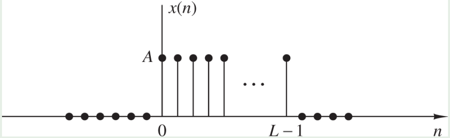

Figure 15: Discrete-time rectangular pulse.

- Fourier transform of this signal is

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = \sum_{n=0}^{L-1} Ae^{-j\omega n} = A\frac{1-e^{-j\omega L}}{1-e^{-j\omega}}$$

Energy Density Spectrum of Aperiodic Signals

Example (continued)

$$X(\omega) = Ae^{-j(\omega/2)(L-1)}\frac{\sin(\omega L/2)}{\sin(\omega/2)}$$

$$|X(\omega)| = \begin{cases} |A|L, & \omega = 0 \\ |A|\left|\frac{\sin(\omega L/2)}{\sin(\omega/2)}\right|, & \text{otherwise} \end{cases}$$

$$\measuredangle X(\omega) = \measuredangle A - \frac{\omega}{2}(L-1) + \measuredangle\frac{\sin(\omega L/2)}{\sin(\omega/2)}$$

Energy Density Spectrum of Aperiodic Signals

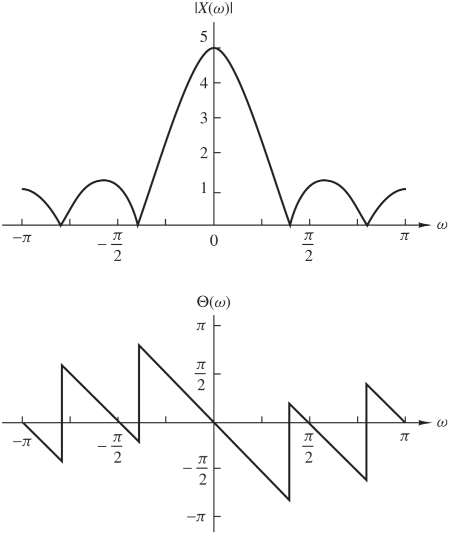

Figure 16: Magnitude and phase of Fourier transform of the discrete-time rectangular pulse in Fig. 15, for the case $A = 1$ and $L = 5$.

References

John G. Proakis, Dimitris G. Manolakis, *Digital Signal Processing: Principles, Algorithms, and Applications*, Prentice Hall, 2006.