# Digital Signal Processing

## Frequency-Domain Analysis of LTI Systems

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# Frequency-Domain Characteristics of LTI Systems

- We develop characterization of LTI systems in frequency domain
    - Basic excitation signals are complex exponentials and sinusoidal functions
    - Characteristics of system are described by a function of $\omega$ called frequency response, which is Fourier transform of impulse response $h(n)$ of system
- Frequency response function completely characterizes an LTI system in frequency domain
    - This allows us to determine steady-state response of system to any arbitrary weighted linear combination of sinusoids or complex exponentials

# The Frequency Response Function

- We know that response of any relaxed LTI system to an arbitrary input signal $x(n)$ is given by convolution sum formula

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

  - System is characterized in time domain by its unit sample response $h(n)$
- To develop a frequency-domain characterization of system, we excite system with complex exponential

$$x(n) = Ae^{j\omega n}, \qquad -\infty < n < \infty$$

  - $A =$ amplitude
  - $\omega =$ any arbitrary frequency confined to $[-\pi, \pi]$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)[Ae^{j\omega(n-k)}] = A\left[\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}\right]e^{j\omega n} \qquad (1)$$

# The Frequency Response Function

- The term in brackets in (1) is Fourier transform of unit sample response $h(k)$ of system

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}$$

Hence response of system to $x(n) = Ae^{j\omega n}$ is

$$y(n) = AH(\omega)e^{j\omega n}$$

  - Response is also in form of a complex exponential with the same frequency as input, but altered by multiplicative factor $H(\omega)$
  - As a result of this characteristic behavior, $x(n) = Ae^{j\omega n}$ is called an **eigenfunction** of system
    - An eigenfunction of a system is an input signal that produces an output that differs from input by a constant multiplicative factor
    - Multiplicative factor (in this case $H(\omega)$) is called an **eigenvalue** of system

# The Frequency Response Function

## Example

- Determine output sequence of system with impulse response

$$h(n) = (\tfrac{1}{2})^n u(n)$$

when input is

$$x(n) = Ae^{j\pi n/2}, \qquad -\infty < n < \infty$$

- Fourier transform of $h(n)$

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} = \frac{1}{1 - \frac{1}{2}e^{-j\omega}} \xrightarrow{\omega=\pi/2} H(\frac{\pi}{2}) = \frac{1}{1 + j\frac{1}{2}} = \frac{2}{\sqrt{5}}e^{-j26.6^\circ}$$

$$y(n) = AH(\omega)e^{j\omega n} = A\left(\frac{2}{\sqrt{5}}e^{-j26.6^\circ}\right)e^{j\pi n/2}$$

$$= \frac{2}{\sqrt{5}}Ae^{j(\pi n/2 - 26.6^\circ)}, \qquad -\infty < n < \infty$$

The only effect of system on input signal is to scale amplitude by $2/\sqrt{5}$ and shift phase by $-26.6^\circ$

## Example (continued)

- If input sequence is

$$x(n) = Ae^{j\pi n}, \qquad -\infty < n < \infty$$

at $\omega = \pi$

$$H(\pi) = \frac{1}{1 - \frac{1}{2}e^{-j\pi}} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$$

and output of system is

$$y(n) = \tfrac{2}{3}Ae^{j\pi n}, \qquad -\infty < n < \infty$$

- If we alter frequency of input signal, effect of system on input also changes and hence output changes
- $H(\pi)$ is purely real
    - Phase associated with $H(\omega)$ is zero at $\omega = \pi$

# The Frequency Response Function

- In general, $H(\omega)$ is a complex-valued function of $\omega$

$$H(\omega) = |H(\omega)| e^{j\Theta(\omega)}$$

  - $|H(\omega)|$ = magnitude of $H(\omega)$
  - $\Theta(\omega) = \measuredangle H(\omega)$, which is phase shift imparted on input signal by system at frequency $\omega$

- Since $H(\omega)$ is Fourier transform of $\{h(k)\}$, $H(\omega)$ is a periodic function with period $2\pi$

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k) e^{-j\omega k}$$

  - Unit impulse $h(k)$ is related to $H(\omega)$ through integral expression

$$h(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\omega) e^{j\omega k} d\omega$$

# The Frequency Response Function

- For an LTI system with a real-valued impulse response, magnitude and phase functions possess symmetry properties

$$
\begin{aligned}
H(\omega) &= \sum_{k=-\infty}^{\infty} h(k) e^{-j\omega k} = \sum_{k=-\infty}^{\infty} h(k)\cos\omega k - j\sum_{k=-\infty}^{\infty} h(k)\sin\omega k \\
&= H_R(\omega) + jH_I(\omega) = |H(\omega)| e^{j\Theta(\omega)} \\
&= \sqrt{H_R^2(\omega) + H_I^2(\omega)}\, e^{j\tan^{-1}[H_I(\omega)/H_R(\omega)]}
\end{aligned}
$$

where

$$
H_R(\omega) = \sum_{k=-\infty}^{\infty} h(k)\cos\omega k \quad \text{and} \quad H_I(\omega) = -\sum_{k=-\infty}^{\infty} h(k)\sin\omega k
$$

  - $H_R(\omega)$: even, $H_I(\omega)$: odd $\longrightarrow$ $|H(\omega)|$: even, $\Theta(\omega)$: odd
    - If we know $|H(\omega)|$ and $\Theta(\omega)$ for $0 \le \omega \le \pi$, we also know these functions for $-\pi \le \omega \le 0$

# The Frequency Response Function

## Example

- Determine magnitude and phase of $H(\omega)$ for three-point **moving average (MA)** system

$$y(n) = \tfrac{1}{3}[x(n+1) + x(n) + x(n-1)]$$

  and plot these two functions for $0 \leq \omega \leq \pi$

- We have

$$h(n) = \{\frac{1}{3}, \underset{\uparrow}{\frac{1}{3}}, \frac{1}{3}\}$$

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} = \frac{1}{3}(e^{j\omega} + 1 + e^{-j\omega}) = \frac{1}{3}(1 + 2\cos\omega)$$

$$|H(\omega)| = \frac{1}{3}|1 + 2\cos\omega| \quad \text{and} \quad \Theta(\omega) = \begin{cases} 0, & 0 \leq \omega \leq 2\pi/3 \\ \pi, & 2\pi/3 \leq \omega < \pi \end{cases}$$

# The Frequency Response Function

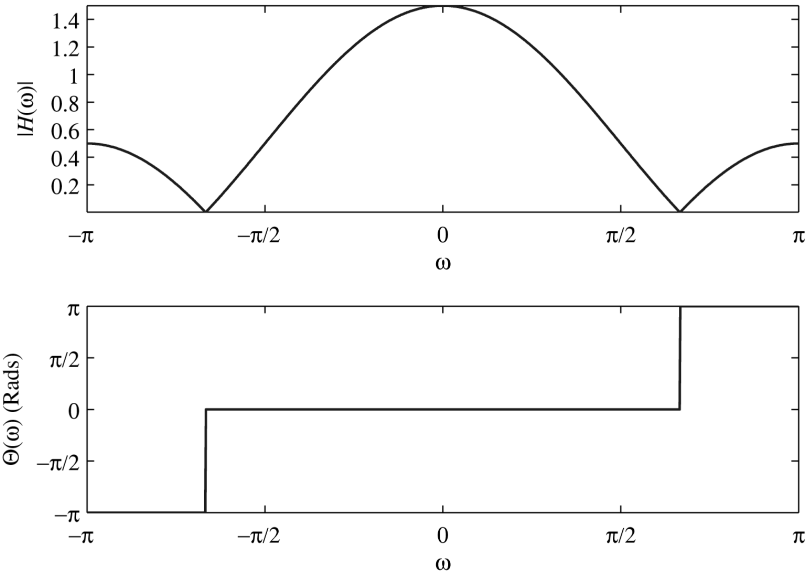


Figure 1: Magnitude and phase responses for the MA system in Example.

# References

John G. Proakis, Dimitris G. Manolakis, *Digital Signal Processing: Principles, Algorithms, and Applications*, Prentice Hall, 2006.