Fixed-Parameter Algorithms, IA166

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Outline

Classical Complexity

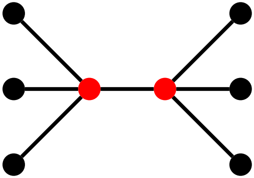
- The complexity of a problem is measured in terms of its **input size**.
- A problem is considered tractable if it can be solved in **polynomial time** and intractable if it is at least NP-hard.
- Unfortunately, many important problems are NP-hard so what can we do to tackle these problems?

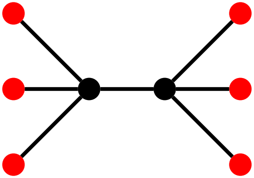
Parameterized Complexity

- The complexity of a problem is measured in terms of its **input size** and any number of additional **parameters.**
- Taking into account additional parameters:
  - provides a more fine-grained view of the complexity of a problem.
  - tells us where the exponential explosion of an NP-hard problem comes from.
  - allows us to design tailored algorithms for different parameterizations.

Parameterized Complexity

Problem:	MINIMUM VERTEX COVER	MAXIMUM INDEPENDENT SET
Input:	Graph G, integer k	Graph G, integer k
Question:	Is it possible to cover the edges with k vertices?	Is it possible to find k independent vertices?





Classical	NP-complete	NP-complete

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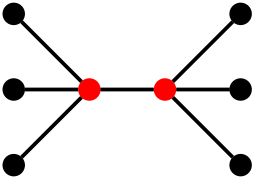

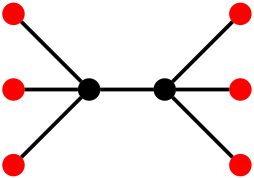

Classical	NP-complete	NP-complete
trivial algorithm	$O(n^k)$	$O(n^k)$

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	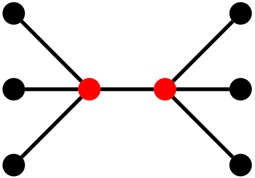	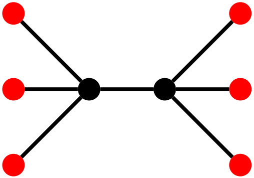
Classical	NP-complete	NP-complete
trivial algorithm	$O(n^k)$	$O(n^k)$
	$O(2^k n^2)$ algorithm exists	No $n^{o(k)}$ algorithm known

A simple algorithm for MINIMUM VERTEX COVER

Input: Graph G and integer k.

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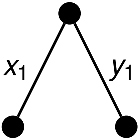

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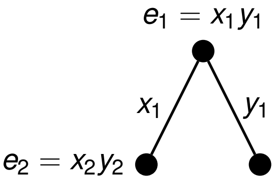

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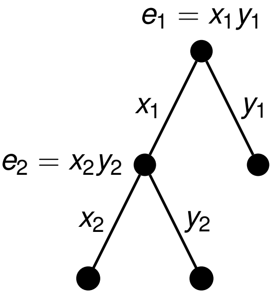

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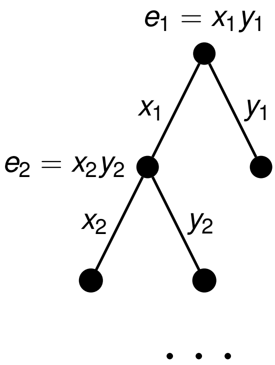

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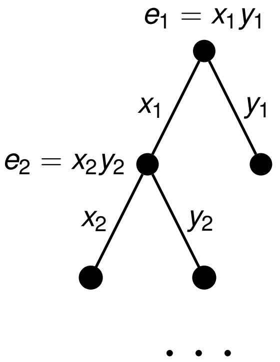

Running time

- at every node there are 2 choices;
- height of the search tree is at most k;
- number of nodes in the search tree is at most 2^k;
- complete search possible in $O(2^k n^c)$

Outline

Fixed-Parameter tractability

Definition

A **parameterization** of a decision problem is a function that assigns an integer parameter (usually denoted by k) to each input instance.

What can the parameter be?

- The size of the solution we are looking for.
- The maximum degree of the input graph.
- The diameter of the input graph.
- The length of clauses in the input SAT-formula.
- ...

Fixed-Parameter Tractability

Definition

A **parameterized problem** is a language $L \subseteq \Sigma^* \times \Sigma^*$ where Σ is a finite alphabet. The first component is the classical input and second component is the **parameter** of the problem.

Definition

A parameterized problem is **fixed-parameter tractable (FPT)** if there is an $f(k)n^c$ time algorithm for an arbitrary function f of the parameter k, input size n, and some constant c.

Fixed-Parameter Tractability

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Example: MINIMUM VERTEX COVER parameterized by the solution size is FPT: we have already seen that it can be solved in time $O(2^k n^2)$.

Better algorithms are known (and are still being developed), e.g., $O(1.2832^k k + kn)$.

Main goal (of parameterized complexity): to find efficient FPT algorithms for NP-hard problems.

Fixed-Parameter Tractability

Definition

A parameterized problem is **fixed-parameter tractable (FPT)** if there is an $f(k)n^c$ time algorithm for an arbitrary function f of the parameter k, input size n, and some constant c.

Remarks

- O^*–notation: $O^*(f(k))$ means $O(f(k)n^c)$ for some constant c.
- unless otherwise stated we always use k to denote the parameter and n to denote the input size.

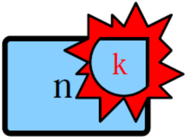
Fixed-Parameter Tractability

Examples of NP-hard problems that are FPT:

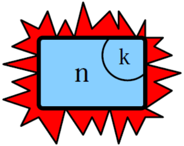
- Finding a vertex cover of size k.
- Finding a path of length k.
- Finding k disjoint triangles.
- Drawing the graph in the plane with k edge crossings.
- Finding disjoint paths that connect k pairs of points.
- ...

Fixed-Parameter Tractability

Fixed-parameter algorithms limit the exponential explosion to the parameter instead of the whole input size.



instead of



- + Guaranteed optimality of the solution.
- + Provable upper bounds on the computational complexity.
- – Exponential running time.

Other approaches to tackle intractable problems:

- Randomized algorithms
- Approximation algorithms
- Heuristics
- Average Case Analysis
- New models of computing (DNA or quantum computing)

Outline

VERTEX COVER an illustrative example

VERTEX COVER **Parameter: k**

Input: An undirected graph $G = (V, E)$ and a natural number k.
Question: Find a subset of vertices $C \subseteq V$ of size at most k such that each edge in E has at least one of its endpoints in C.

Solution methods:

- Bounded Search Tree: $O^*(1.28^k)$.
- Data reduction by preprocessing: techniques by Buss, Nemhauser Trotter.

VERTEX COVER an illustrative example

- Parameterizing
- Specializing
- Generalizing
- Counting or Enumeration
- Lower bounds
- Implementing and applying
- Exploiting the structure given by a VERTEX COVER for other problems

VERTEX COVER an illustrative example

- Parameterizing
  - Size of the vertex cover;
  - Dual parameterization: INDEPENDENT SET;
  - Parameterizing above guaranteed values, e.g., in planar graphs;
  - Structure of the input graph, e.g., treewidth
- Specializing
- Generalizing
- Counting or Enumeration
- Lower bounds
- Implementing and applying
- Exploiting the structure given by a VERTEX COVER for other problems

VERTEX COVER an illustrative example

- Parameterizing
- Specializing: special graph classes, e.g., planar graphs $O^*(c^{\sqrt{k}})$.
- Generalizing
- Counting or Enumeration
- Lower bounds
- Implementing and applying
- Exploiting the structure given by a VERTEX COVER for other problems

VERTEX COVER an illustrative example

- Parameterizing
- Specializing
- Generalizing: WEIGHTED VERTEX COVER, CAPACITATED VERTEX COVER, HITTING SET, ...
- Counting or Enumeration
- Lower bounds
- Implementing and applying
- Exploiting the structure given by a VERTEX COVER for other problems

VERTEX COVER an illustrative example

- Parameterizing
- Specializing
- Generalizing
- Counting or Enumeration:
  - Counting: $O^*(1.47^k)$.
  - Enumeration: $O^*(2^k)$.
- Lower bounds
- Implementing and applying
- Exploiting the structure given by a VERTEX COVER for other problems

VERTEX COVER an illustrative example

- Parameterizing
- Specializing
- Generalizing
- Counting or Enumeration
- Lower bounds: widely open!
- Implementing and applying
- Exploiting the structure given by a VERTEX COVER for other problems

VERTEX COVER an illustrative example

- Parameterizing
- Specializing
- Generalizing
- Counting or Enumeration
- Lower bounds
- Implementing and applying: re-engineering case distinctions, parallelization, ...
- Exploiting the structure given by a VERTEX COVER for other problems

VERTEX COVER an illustrative example

- Parameterizing
- Specializing
- Generalizing
- Counting or Enumeration
- Lower bounds
- Implementing and applying
- Exploiting the structure given by a VERTEX COVER for other problems: solve related problems using an optimal vertex cover.

Outline

The Art of Problem Parameterization

- Parameter really small?
- Guaranteed parameter value?
- More than one obvious parameterization?
- Close to “trivial” problem instances?

Outline

Algorithmic Techniques

Powerful toolbox for designing FPT algorithms with significant advances over the last 20 years:

Bounded Search Tree

Kernelization

Color Coding

Treewidth

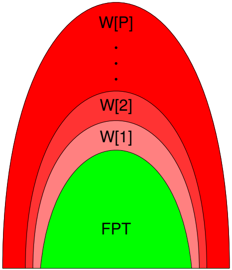
Integer Linear Programming

Iterative Compression

Outline

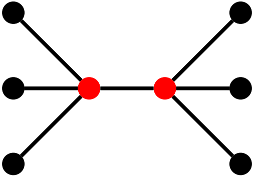
Fixed-Parameter Intractability

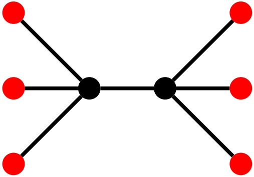
- Corresponding to the class NP in the classical setting in parameterized complexity there is a whole hierarchy of complexity classes (the **W-hierarchy**).
- All problems that are at least W[1]**-hard** are considered **fixed-parameter intractable.**
- Most natural problems are either FPT, W[1]-complete or W[2]-complete.

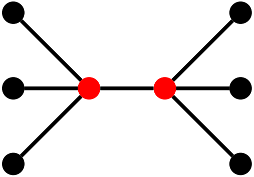


Parameterized Complexity Classes

Fixed-Parameter Intractability







VERTEX COVER	INDEPENDENT SET	DOMINATING SET
NP-complete	NP-complete	NP-complete
FPT	W[1]**-complete**	W[2]**-complete**

Literature

- **Rolf Niedermeier, Invitation to Fixed-Parameter Algorithms, Oxford University Press 2006**
- Joerg Flum and Martin Grohe, Parameterized Complexity Theory, Springer 2006
- Micheal R. Fellows and Rodney G. Downey, Parameterized Complexity, Springer 1999