Fixed-Parameter Algorithms, IA166

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Outline

Iterative Compression

Example: Directed Feedback Vertex Set

k-DIRECTED FEEDBACK VERTEX SET **Parameter:** k

Input: A digraph D and an integer k.
Question: Is there a set $S \subseteq V(D)$ with $|S| \leq k$ and $D \setminus S$ is a DAG, i.e., a directed acyclic graph?

Example: Directed Feedback Vertex Set

Using the sequence $D_i := D[v_1, \ldots, v_i]$ for an arbitrary ordering $v_1, \ldots, v_{|V(D)|}$ of the vertices of D we obtain:

(1) $S_1 := \emptyset$ is a k-DFVS for D_1.

(2) If S_i is a k-DFVS for D_i then $S_{i+1} := S_i \cup \{v_{i+1}\}$ is a $(k+1)$-DFVS for D_{i+1}.

(3) If D_i has no k-DFVS then D has no k-DFVS.

(4) FPT-algorithm for compression version????

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Hence, we only need to find an FPT-algorithm for the compression version of the problem!

Example: Directed Feedback Vertex Set

Answer

Again we guess the intersection of S with an optimal solution. There are 2^{k+1} such guesses and for each guess we have to solve an instance of k-S-DISJOINT DIRECTED FEEDBACK VERTEX SET defined below.

k-S-DISJOINT DIRECTED FEEDBACK VERTEX SET **Parameter:** $|S|$

Input: A digraph D , and a DFVS S of D.
Question: Is there a $(|S| - 1)$-DFVS for D that is disjoint from S?

Example: Directed Feedback Vertex Set

k-S-DISJOINT FEEDBACK VERTEX SET **Parameter: $|S|$**

Input: A graph D , and a FVS S of D.
Question: Is there a $(|S| - 1)$-FVS for D that is disjoint from S?

Problem

We need a novel approach here as the kernelization approach used for k-S-DFVS fails!

Example: Directed Feedback Vertex Set

Topological Ordering

The vertices of a DAG, i.e., directed acyclic graph, can be ordered $v_1, \ldots, v_n$ such that for every $1 \leq i < j \leq n$, there is no arc from the vertex v_j to the vertex v_i. Such an ordering is often called a topological ordering and can be found in polynomial time.

Example: Directed Feedback Vertex Set

Reduction of k-S-DDFVS to the k-SKEW SEPARATOR PROBLEM.

MINIMUM SKEW SEPARATOR (k-MSS) **Parameter: k**

Input: Digraph G, vertex sequences $S = (s_1, \ldots, s_l)$ and $T = (t_1, \ldots, t_l)$ where all s_i have in-degree 0 and all t_i have out-degree 0, and an integer k.

Question: Does G have a skew-separator (SS) C of size at most k, i.e., is there a $C \subseteq V(G) \setminus (S \cup T)$ such that $|C| \leq k$ and for all $j \leq i$, t_j is not reachable from s_i in $G \setminus C$?

Example: Directed Feedback Vertex Set

Definition

Let (D, S, k) be an instance of k-S-DDFVS and let $\sigma : \{1, \ldots, k+1\} \to S$ be a bijective function (a numbering of S). We define the graph $D(\sigma)$ to be the graph obtained from $D \setminus S$ and for every $1 \leq i \leq k+1$:

- adding the vertices s_i and t_i and
- for every out-neighbor v of $\sigma(i)$ adding an arc (s_i, v), and
- for every in-neighbor v of $\sigma(i)$ adding an arc (v, t_i)

Example: Directed Feedback Vertex Set

Lemma

$S' \subseteq V \setminus S$ is a k-S-DDFVS iff there is a numbering σ of S such that S' is a k-Skew-separator for $D(\sigma)$.

Proof:

$(\rightarrow)$ Let S' be a k-S-DDFVS for D. Then $D \setminus S'$ is acyclic and admits a topological ordering $v_1, \ldots, v_m$.

All vertices of S are part of $D \setminus S'$. Hence, we can use the ordering to define a numbering σ of S such that for all $j < i$, $\sigma(j)$ is not reachable from $\sigma(i)$.

Example: Directed Feedback Vertex Set

Lemma

$S' \subseteq V \setminus S$ is a k-S-DDFVS iff there is a numbering σ of S such that S' is a k-Skew-separator for $D(\sigma)$.

Proof, continued:

Consider $D(\sigma)$ and suppose that $D(\sigma) \setminus S'$ contains an (s_i, t_j)-path for $j \leq i$.

If $j < i$ this gives a $(\sigma(i), \sigma(j))$-path in $D \setminus S'$ contradicting the choice of σ.

If $j = i$, this gives a cycle in $G \setminus S'$ (containing $\sigma(i)$), a contradiction.

Hence, S' is a Skew-Separator for $D(\sigma)$.

Example: Directed Feedback Vertex Set

Lemma

$S' \subseteq V \setminus S$ is a k-S-DDFVS iff there is a numbering σ of S such that S' is a k-Skew-separator for $D(\sigma)$.

Proof, continued:

$(\leftarrow)$ Let S' be a k-Skew-Separator for $D(\sigma)$ and suppose that $G \setminus S'$ contains a cycle C.

Because S is a DFVS for D, C contains at least one vertex from S.

If C contains exactly one vertex $\sigma(i)$ from S, then C corresponds to an (s_i, t_i)-path in $D(\sigma) \setminus S'$, a contradiction.

Example: Directed Feedback Vertex Set

Lemma

$S' \subseteq V \setminus S$ is a k-S-DDFVS iff there is a numbering σ of S such that S' is a k-Skew-separator for $D(\sigma)$.

Proof, continued:

If C contains $l \geq 2$ vertices from S, then let $\sigma(i_0), \sigma(i_1), \ldots, \sigma(i_{l-1})$ be the order of those vertices in C. Note that i_x and i_y are distinct when $x \neq y$. Hence, there is an x such that $i_x > i_{x+1 \mod l}$. Let $i = i_x$ and $j = i_{x+1 \mod l}$.

The subpath of C from $\sigma(i)$ to $\sigma(j)$ corresponds to an (s_i, t_j)-path in $D(\sigma) \setminus S'$ with $j < i$, a contradiction. □

Example: Directed Feedback Vertex Set

Recall:

k-MINIMUM SKEW SEPARATOR can be solved in time $4^k n^{O(1)}$.

Corollary

k-S-DDFVS can be solved in time $(k+1)!4^k n^{O(1)}$.

Hence, the combined parameter function for k-DFVS is:

$$\sum_{l=0}^{k} \binom{k+1}{l+1}(l+1)!4^l =$$
$$\sum_{l=0}^{k} \frac{(k+1)!}{(k-l)!}4^l < (k+1)!(k+1)4^k$$

Theorem

k-S-DFVS can be solved in time $O^*(4^k(k+1)!)$.