Fixed-Parameter Algorithms, IA166

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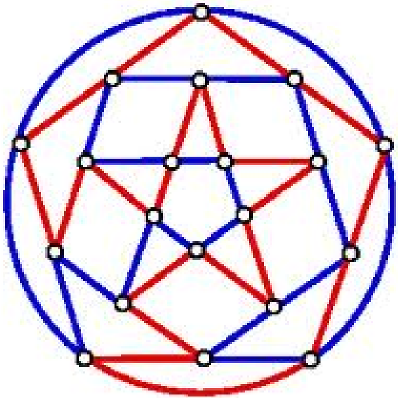
Outline

Planar Graphs



Definitions and Basic Facts

Let G be a graph.

- A drawing of G in the plane $\mathbb{R}^2$ is a mapping Π that maps all vertices $v \in V(G)$ to distinct points $\Pi(v)$ in $\mathbb{R}^2$, and edges $\{u, v\} \in E(G)$ to simple curces between $\Pi(u)$ and $\Pi(v)$.
- A planar embedding of G is a drawing of G without edge crossings, i.e., the curces corresponding to the 2 edges can only have a common endpoints of the edges in common.
- A plane graph (G, Π) consists of G and a planar embedding Π of G.
- G is planar if it admits a planar embedding.

Definitions and Basic Facts

Let G be a graph.

- Let Π be a planar embedding of G. The faces of Π are the maximal connected subsets of $\mathbb{R}^2$ that contain no images of Π, i.e., the regions of $\mathbb{R}^2 \setminus \Pi(V \cup E)$.
- A plane graph has 1 unbounded face. This is called the outer face.

Proposition

Let (G, Π) be a connected plane graph such that every edge lies on a cycle of G. Then the boundaries of faces are (images of) cycles, and (the image of) every edge is contained in the boundary of two faces.

Definitions and Basic Facts

Let G be a graph.

Euler's formula

Let (G, Π) be a non-empty connected plane graph with n vertices, m edges and f faces. Then $n - m + f = 2$.

Definitions and Basic Facts

Let (G, Π) be a plane graph. A triangulation of (G, Π) is a plane graph (G', Π') with $V(G) = V(G')$, $E(G) \subseteq E(G')$, and Π' extends Π such that

- G' is connected and every edge of G' lies on a cycle, and
- all faces of (G', Π') are triangles.

Proposition

If $|V(G)| \geq 3$, a triangulation of (G, Π) exists and can be constructed in time $O(|E(G)|)$.

Definitions and Basic Facts

Proposition

Let G be a planar graph with $|V(G)| \geq 3$. Then $|E(G)| \leq 3|V(G)| - 6$.

Proof:

Let $n := |V(G)|$ and $m = |E(G)|$. Let Π be a planar embedding of G with f faces. By the previous proposition it suffices to show the statement in case (G, Π) is a triangulation.
In this case all faces are triangles and every edge is part of 2 faces, hence $3f = 2m$.
Then Euler's formula gives $m = n + f - 2 = n + \frac{2}{3}m - 2$ and $m = 3n - 6$. □

Definitions and Basic Facts

Corollary

Every planar graph has a vertex of degree at most 5.

Theorem

In linear time it can be checked whether a given graph is planar and if so a planar embedding can be computed.

Four Color Theorem

Every planar graph admits a proper 4-vertex coloring.

Outline

k-Planar Independet Set

k-Planar Independet Set **Parameter: k**

Input: A planar graph G and an integer k.
Question: Does G have an independent set of size at least k?

For the non-planar version of the problem, FPT algorithms are unlikely to exists (W[1]-hard), but for the planar version FPT algorithms are easily found.

k-PLANAR INDEPENDET SET

Trivial FPT algorithms for k-PLANAR INDEPENDENT SET:

Kernelization

Because of the Four Color Theorem G is 4-colorable. Hence, G has an independent set of size at least $|V(G)|/4$.

Hence, without any preprocessing, a $4k$-vertex kernel is obtained, which is actually also a $4k$-edge kernel because $|E(G)| \leq 3|V(G)| - 6$.

k-PLANAR INDEPENDET SET

Trivial FPT algorithms for k-PLANAR INDEPENDENT SET:

Branching

Consider a vertex v of degree at most 5. A maximal independent set contains v or 1 of its neighbors.

Branching on this choice yields a search tree with at most 6^k leaves.

Treewidth of Planar Graphs

Some Definitions:

- The length of a (v_1, v_k)-path $v_1, \ldots, v_k$ is $k - 1$, and the distance between 2 vertices u and v is the minimum length over all (u, v)-paths, or ∞ is no such path exists.
- The diameter of a graph is the maximum distance between any two vertices.
- The height of a rooted tree is the maximum distance from the root to a leaf.

Treewidth of Planar Graphs

Theorem

Let G be a planar graph for which a rooted spanning tree T of height l is given. Then a tree decomposition of G of width at most $3l$ exists, and can be constructed in polynomial time.

Corollary

A planar graph with diameter D has a tree decomposition of width at most $3D$.

Proof: Construct a breadth-first search tree starting at arbitrary root vertex. □

Treewidth of Planar Graphs

Theorem

Let G be a planar graph for which a rooted spanning tree T of height l is given. Then a tree decomposition of G of width at most $3l$ exists, and can be constructed in polynomial time.

Proof:

Let (G, Π) be a planar embedding of G and T be the spanning tree of height l with root r. W.l.o.g. we can assume that G is triangulated.

We may assume that $|V(G)| \geq 4$ (the case $|V(G)| \leq 3$ is trivial). Hence, 2 faces share at most 1 edge.

Treewidth of Planar Graphs

Proof, continued:

Let F be the set of faces of (G, Π). Let T^* be the graph with vertex set $V(T^*) := F$ and $\{f, g\} \in E(T^*)$ iff the boundaries of the faces f and g share an edge in $E(G) \setminus E(T)$.

For $f \in F$, define the bag X_f to contain the 3 vertices u, v, w on the boundary of f, and all of their ancestors with respect to T and r.

We will prove that (T^*, X) is the desired tree decomposition of G.

Lemma

(T^*, X) is a tree decomposition of G of width at most $3l$.

Treewidth of Planar Graphs

Claim

T^* contains no cycles.

Proof:

A cyle $C := f_0, \ldots, f_k, f_0$ of T^* corresponds to a simple closed curve C in the plane through the faces $f_1, \ldots, f_k$ that crosses the edge shared by f_i and $f_{i+1 \mod k}$ exactly once for all i and crosses no other edges.

By the Jordan Curve Theorem C divides the plane into 2 regions, which both contain at least 1 vertex.
Because C crosses no edges of T, this contradicts that T is a spanning tree. □

Treewidth of Planar Graphs

Claim

For every face f, $|X_f| \leq 3l + 1$.

Proof:

X_f contains the 3 vertices on its boundary and all of its ancestors in T.
Because T has height l, every vertex has at most l ancestors.
The root r is shared a shared ancestor of the 3 vertices. Hence,
$|X_f| \leq 3 + 3l - 2 = 3l + 1$. □

Treewidth of Planar Graphs

Claim

For every edge $\{u, v\} \in E(G)$ there is an $f \in V(T^*)$ with $\{u, v\} \in X_f$.

Proof:

This is trivial because every edge lies on the boundary of at leasy one face. □

Treewidth of Planar Graphs

Claim

For every $v \in V(G)$, the subgraph of T^* induced by $X^{-1}(v)$ is non-empty and connected.

Proof:

By induction over the height of the subtree rooted at v.
Induction Start: If v is a leaf of T, then $v \in X_f$ iff v is incident with f. Because v is a leaf, the faces incident with v induce a path in T^*.
Induction Step: Suppose v is not a leaf and $v \neq r$. Let $v_0, \dots, v_{d-1}$ be the neighbors of v in clockwise order around v such that v_0 is the parent of v in T.
Let $f_0, \dots, f_{d-1}$ be the faces incident with v such that f_i is incident with v_i and $v_{i+1 \mod d}$.

Treewidth of Planar Graphs

Proof, continued:

Let $f_0, \ldots, f_{d-1}$ be the faces incident with v such that f_i is incident with v_i and $v_{i+1 \mod d}$.
Let $v_{i_1}, \ldots, v_{i_k}$ be the children of v in T. Then v is contained in all bags X_{f_i} and in all bags that also contain a child v_{i_j}, but in no other bags, i.e.:

$$X^{-1}(v) = \{f_0, \ldots, f_{d-1}\} \cup X^{-1}(v_{i_1}) \cup \cdots \cup X^{-1}(v_{i_k})$$

By induction $X^{-1}(v_{i_j})$ is connected for every j.
If the edge shared by f_i and f_{i+1} is not in T, then they are adjactent in T^*. Otherwise, they share an edge $\{v, v_{i_j}\}$, and are both part of the connected set $X^{-1}(v_{i_j})$.
This shows that $X^{-1}(v)$ is connected in T^*. If v is the root of T the argument is similar. □

Treewidth of Planar Graphs

Because the root r is part of every bag X_f and $X^{-1}(r)$ induces a connected subgraph of T^* by the previous claim it follows that T^* is also connected.

Summary:

- T^* contains no cycles and is connected.
- For every f, $|X_f| \leq 3l + 1$.
- Every edge $\{u, v\} \in E(G)$ is covered by some X_f.
- For every $v \in V(G)$ the subgraph of T^* induced by $X^{-1}(v)$ is connected.

Hence, (T^*, X) is the desired tree decomposition of G. □

k-Planar Dominating Set

k-Planar Dominating Set **Parameter: k**

Input: A planar graph G and an integer k.
Question: Does G have a dominating set S of cardinality at most k?

Theorem

k-Planar Dominating Set is fixed parameter tractable.

k-PLANAR DOMINATING SET

Theorem

k-PLANAR DOMINATING SET is fixed parameter tractable.

Proof:

W.l.o.g. we can assume that G is connected. Compute the diameter d of G in polynomial time (e.g. using BFS trees). If $d \geq 3k$ then return NO. This is correct because a vertex can dominate at most 3 vertices of any shortest path. Otherwise, planarly embed the graph, construct a BFS tree of height at most $3k - 1$, and use it to construct a tree decomposition of width at most $3(3k - 1)$ (all can be done in polynomial time). Use dynamic programming to find the correct answer. □

Summary and Outlook

- When restricted to planar graphs, FPT algorithms exist for problems that are unlikely to admit FPT algorithms for general graphs (e.g. k-INDEPENDENT SET and k-DOMINATING SET).
- One essential property for this is that for planar graphs, the treewidth is bounded by a function of the diameter (they have bounded local treewidth). There are many more graph classes with bounded local treewidth, and this can be used to construct FPT algorithms for them.

Outline

Locally Bounded Treewidth

Definition

Let $\mathcal{C}$ be a class of graphs. $\mathcal{C}$ has locally bounded treewidth if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for every $G \in \mathcal{C}$, $v \in V(G)$, and natural number r it holds that $\text{tw}(G[N_r^G[v]]) \leq f(r)$.

- Every class of graphs of bounded treewidth also has locally bounded treewidth.
- We have already seen that planar graphs have locally bounded treewidth.
- There are many more important graph classes that have locally bounded treewidth such as graph classes of bounded degree, graph classes of bounded genus, etc..

A Meta-Theorem for FO-Logic and Locally Bounded Treewidth

Theorem

Let $\mathcal{C}$ be a class of graphs with locally bounded treewidth and Φ be an FO-formula of length k. Then it can be decided in time $f(k)O(n^2)$ whether $G \models \Phi$ for every $G \in \mathcal{C}$.

- FO-definable problems include problems such as k-DOMINATING SET and k-INDEPENDENT SET
- it does not include MSO-definable problems such as COLORING and HAMILTONICITY, etc.

A Meta-Theorem for FO-Logic and Locally Bounded Treewidth

To sketch a proof of the Meta-Theorem we need the following Notions and Facts:
Let r be a natural number.

- We denote by $d(x, y) > r$ the FO-formula such that $G \models d(v, u) > r$ iff the vertices v and u have distance at least r in G.
- We say a FO-formula $\Phi(x)$ is r-local iff the validity of $\Phi(x)$ only depends on the r-neighborhood of x, i.e., if for all graphs G and vertices $v \in V(G)$ it holds that $G \models \Phi(v)$ iff $G[N_r^G[v]] \models \Phi(v)$.

A Meta-Theorem for FO-Logic and Locally Bounded Treewidth

Gaifman's Theorem

Every FO-sentence is equivalent to a Boolean combination of sentences of the form:

$$\exists x_1, \dots x_l (\bigwedge_{1 \le i < j \le l} d(x_i, x_j) > 2r \wedge \bigwedge_{1 \le i \le l} \Phi(x_i))$$

with $l, r \geq 1$ and r-local $\Phi(x)$. Furthermore, such a boolean combination can be found in an effective way.

The above theorem is sometimes also called the Locality Theorem for FO-Logic.

A Meta-Theorem for FO-Logic and Locally Bounded Treewidth

Let G be a graph, $S \subseteq V(G)$ and $l, r \in \mathbb{N}$. Then S is (l, r)-scattered if there exist $v_1, \ldots, v_l \in S$ such that $d_G(v_i, v_j) > r$ for every $1 \leq i < j \leq l$.

Lemma

Let $\mathcal{C}$ be a class of graphs of locally bounded treewidth. Then there is an algorithm that, given $G \in \mathcal{C}$, a set $S \subseteq V(G)$ and $l, r \in \mathbb{N}$, decides if S is (l, r)-scattered in time $g(l, r)|V(G)|$.

A Meta-Theorem for FO-Logic and Locally Bounded Treewidth

Lemma

Let $\mathcal{C}$ be a class of graphs of locally bounded treewidth. Then there is an algorithm that, given $G \in \mathcal{C}$, a set $S \subseteq V(G)$ and $l, r \in \mathbb{N}$, decides if S is (l, r)-scattered in time $g(l, r)|V(G)|$.

Proof:

We start by computing a maximal set $T \subseteq S$ such $d_G(t_i, t_j) > r$ for every $1 \leq i < j \leq |T|$. Clearly, such a set T can be easily found by a simple greedy algorithm. If $|T| \geq l$ then we are done. So suppose $|T| < l$. Because of the maximality of T it holds that $S \subseteq N_r^G[T]$ and S is (l, r)-scattered in G iff S is (l, r)-scattered in $N_{2r}^G[T]$.

A Meta-Theorem for FO-Logic and Locally Bounded Treewidth

Lemma

Let $\mathcal{C}$ be a class of graphs of locally bounded treewidth. Then there is an algorithm that, given $G \in \mathcal{C}$, a set $S \subseteq V(G)$ and $l, r \in \mathbb{N}$, decides if S is (l, r)-scattered in time $g(l, r)|V(G)|$.

Proof:

We now show that the treewidth of $G[N_{2r}^{G}[T])]$ is bounded by some function that depends only on l and r. Using Courcelle's Theorem this implies the lemma. To see this note that the diameter of every component of $N_{2r}^{G}[T]$ is bounded by $(4r+1)l$ and hence every such component is contained in the $(4r+1)l$ neighborhood of any vertex in that component. □

A Meta-Theorem for FO-Logic and Locally Bounded Treewidth

Theorem

Let $\mathcal{C}$ be a class of graphs with locally bounded treewidth and Φ be an FO-formula of length k. Then it can be decided in time $f(k)O(n^2)$ whether $G \models \Phi$ for every $G \in \mathcal{C}$.

Proof:

Let Φ be the given FO-formula of length at most k and $G \in \mathcal{C}$. Because of Gaifman's Theorem we can assume that Φ has the form:

$$\Phi := \exists x_1, \ldots x_l (\bigwedge_{1 \leq i < j \leq l} d(x_i, x_j) > 2r \wedge \bigwedge_{1 \leq i \leq l} \phi(x_i))$$

with $l, r \geq 1$ and r-local $\phi(x)$.

A Meta-Theorem for FO-Logic and Locally Bounded Treewidth

Theorem

Let $\mathcal{C}$ be a class of graphs with locally bounded treewidth and Φ be an FO-formula of length k. Then it can be decided in time $f(k)O(n^2)$ whether $G \models \Phi$ for every $G \in \mathcal{C}$.

Proof, continued:

Because of Courcelle's Theorem and the fact that $\mathcal{C}$ has bounded local treewidth can decide whether $G \models \phi(v)$ in time $f(k)|V(G)|$ for every $v \in V(G)$. Consequently, we can compute the set $\{ v \in V(G) : G \models \phi(v) \}$ in time $f(|\phi|)|V(G)|^2$. Now, $G \models \phi$ iff S is (l, r)-scattered. Using the previous Lemma it follows that we can decide whether S is (l, r)-scattered in time $g(k)|V(G)|$. This shows the theorem. □

Outline

Outerplanar Graphs and Layers

Let G be a plane graph.

- G is outerplanar or 1-outerplanar if every vertex is incident with the outer face.
- G is k-outerplanar for $k \geq 2$ if deleting all vertices that are incident with the outer face yields a $(k-1)$-outerplanar graph.
- Layer Decomposition: The vertices of a k-outerplanar graph can be partitioned into k layers $L_1, \ldots, L_k$ as follows: L_1 consists of the vertices incident with the outer face, and L_i consists of the vertices incident with the outer face after deleting the vertex sets $L_1, \ldots, L_{i-1}$.

Outerplanar Graphs and Layers

Proposition (1)

Let $L_1, \ldots, L_k$ be a layer decomposition of a k-outerplanar graph (G, Π), and let $L = L_i \cup \cdots \cup L_{i+j}$. A tree decomposition of $G[L]$ of width $3j + 3$ can be found in polynomial time.

Proof:

Add a single vertex r drawn in the outer face of $G[L]$ and connect it to every vertex in L_i while maintaining a plane graph. Add edges to ensure that every vertex in layer L_x has a neighbor in layer L_{x-1} while maintaining a plane graph. Call the resulting plane graph G'. Then a BFS tree of G' rooted at r has height $j + 1$, hence $\mathrm{tw}(G) \leq \mathrm{tw}(G') \leq 3j + 3$. □

Outerplanar Graphs and Layers

Proposition (2)

Let $S_1, \ldots, S_l$ be disjoint vertex sets of G and $S := S_1 \cup \cdots \cup S_l$ such that:

- $\mathrm{tw}(G \setminus S) \leq t$,
- Every component of $G \setminus S$ only has neighbors in S_i and S_{i+1} for some i,
- there are no edges between S_i and S_j if $|j - i| \geq 2$, and
- $|S_i| \leq x$ for every i.

Then $\mathrm{tw}(G) \leq t + 2x$.

Sets $S_1, \ldots, S_l$ that satisfy the above properties are called *t-x-separators*.

Outerplanar Graphs and Layers

Proof:

Construct a tree decomposition as follows: Start with a path on vertices $v_1, \ldots, v_{l-1}$ and let $X(v_i) := S_i \cup S_{i+1}$.
For every component C of $G \setminus S$ that only has neighbors in S_i and S_{i+1}, add a tree decomposition of width t of C, add $S_i \cup S_{i+1}$ to all bags, and connect this tree to v_i with an arbitrary edge. This yields a tree decomposition of width $t + 2x$. □

Outerplanar Graphs and Layers

Theorem

A planar graph G on n vertices has $\mathrm{tw}(G) < 4.9\sqrt{n}$.

Proof:

Consider a planar embedding of G and let k be its outerplanarity. Construct a layer decomposition $L_1, \ldots, L_k$. Let $\alpha = \sqrt{\frac{3}{2}} < 1.225$. Construct t-x-separators $S_1, \ldots, S_l$ with $t = \frac{3}{\alpha}\sqrt{n}$ and $x = \alpha\sqrt{n}$ as follows:

Outerplanar Graphs and Layers

Theorem

A planar graph G on n vertices has $\mathrm{tw}(G) < 4.9\sqrt{n}$.

Proof:

Consider the layers $L_1, \ldots, L_k$ in order. Whenever $|L_i| \leq x$, this L_i is chosen as the next S_j.
Suppose b layers are not selected as separator. Then
$n \geq bx = b\alpha\sqrt{n}$, so $b \leq \sqrt{n}/\alpha$.
Therefore, $\mathrm{tw}(G \setminus S) \leq 3b \leq \frac{3}{\alpha}\sqrt{n}$ by Proposition 1.
Then by Proposition 2,
$\mathrm{tw}(G) \leq t + 2x \leq \frac{3}{\alpha}\sqrt{n} + 2\alpha\sqrt{n} = 4\alpha\sqrt{n} < 4.9\sqrt{n}$. □

Some simple Applications

The following algorithm decides in time $2^{O(\sqrt{k})}n^{O(1)}$ whether a planar graph G on n vertices admits a k-vertex cover:

(1) In polynomial time reduce (G, k) to an equivalent (planar!) instance (G', k') with $n = |V(G')| \leq 2k$ (See the kernelization lecture and note that the reduction rules preserve planarity).

(2) Use the previous theorem to construct a tree decomposition of G' of width $w \in O(\sqrt{n}) = O(\sqrt{k})$.

(3) Use dynamic programming to decide whether G' has a k'-VC in time $2^{O(\sqrt{k})}n^{O(1)}$ (see lecture on dynamic programming over tree decompositions).

Similarly, a $2^{O(\sqrt{k})}n^{O(1)}$ algorithm can be given for k-PLANAR INDEPENDENT SET because we have a $4k$-vertex kernel (on planar graphs) and a $2^{w}n^{O(1)}$ dynamic programming algorithm from a previous lecture.

Advanced Applications

Recall that we had a $5k$-vertex kernel for k-MAX LEAVES SPANNING TREE which used planarity preserving reduction rules.

Question

Can a $2^{O(\sqrt{k})} n^{O(1)}$ algorithm for k-PLANAR MAX LEAVES SPANNING TREE be given?

Answer

Yes, but in this case a $2^{O(w)} n^{O(1)}$ dynamic programming algorithm is far from trivial: such algorithms make heavy use of planarity!

Advanced Applications

Question

Can this approach be used to give a fast FPT algorithm for planar problems without linear kernels?

Answer

Yes, by constructing the separators $S_1, \ldots, S_l$ more smartly, and bounding their size in terms of an optimal solution.

Outline

Grid Minors and Treewidth – General Graphs

- Recall: $G_{k \times k}$ denotes the $k \times k$ grid, which is a planar graph with tree width k.
- Recall: Graph H is a minor of graph G if H can be obtained from G by vertex deletions, edge deletions, and edge constractions. In that case $\mathrm{tw}(H) \leq \mathrm{tw}(G)$.
- Hence, if G has a $G_{k \times k}$ as a minor, then $\mathrm{tw}(G) \geq k$.

Theorem

Every graph of tree width at least $w(k) := 20^{2k^5}$ has $G_{k \times k}$ as a minor.

Grid Minors and Treewidth – General Graphs

Theorem

Let G be a graph that has a $G_{k \times k}$ as a minor. Then G has a k^2-path.

An FPT-algorithm for k-PATH

- Decide whether $\mathrm{tw}(G) \leq w(\sqrt{k})$ and if so construct a tree decomposition.
- Use the tree decompostion to decide whether G has a k-path using an $f(w(\sqrt{k}))n^{O(1)}$ dynamic programming algorithm. (which exists due to Courcelle's Theorem).
- Otherwise, i.e., if $\mathrm{tw}(G) \geq w(\sqrt{k})$ then return YES (This is correct by the previous theorem).

This is by far the most unpractical and slowest FPT algorithm that we have seen yet!

Bidimensionality – the General Framework

The above scheme suggests that in order to prove that a problem admits an FPT algorithm, we only need to show:

(1) For graphs with large grid minors the answer is trivially YES or NO.

(2) The problem can be expressed in MSOL or otherwise solved efficiently on graphs of bounded treewidth.

- For many problems Properites (1) and (2) can be easily verified (e.g., k-MLST, k-FVS, k-VC).
- Not surprisingly, Property (1) above does not hold for problems such as k-INDEPENDENT SET or k-DOMINATING SET.
- Next: For planar and related graph classes the above scheme gives fast and practical FPT algorithm even for k-INDEPENDENT SET and k-DOMINATING SET.

Bidimensionality for Planar Graphs

Theorem

Every planar graph of treewidth at least $6k - 5$ has a $G_{k \times k}$ as a minor.

Theorem

Let G be a planar graph. In polynomial time, a tree decomposition of G of width at most $\frac{3}{2}\mathrm{tw}(G)$ can be constructed, i.e, treewidth is constant factor approximable on planar graphs.

Bidimensionality for Planar Graphs

Suppose that for a parameterized planar graph problem the following properties hold:

(A) for graphs with $G_{c\times c}$ minor the answer is trivially YES or NO, where $c \in O(\sqrt{k})$, and

(B) When a tree decomposition of width w is given, the problem can be solved in time $2^{O(w)}n^{O(1)}$.

Then the following algorithm is a $2^{O(\sqrt{k})}n^{O(1)}$ FPT algorithm:

(1) In polynomial time, compute a $3/2$-approximate tree decomposition (T, X) of G.

(2) If the width of (T, X) is at least $O(\sqrt{k})$, then return the trivial answer.

(3) If the width of (T, X) is at most $O(\sqrt{k})$, then solve the problem by dynamic programming.

Problems that satisfy Property (A) are called bidimensional.

k-VERTEX COVER is bidimensional

Proposition

If a graph G contains a $G_{k\times k}$ as a minor, then G has no vertex cover smaller than $k(k-1)/2$.

Theorem

k-PLANAR VERTEX COVER can be solved in time $O(2^{O(\sqrt{k})}n^{O(1)})$.

Contraction Bidimensionality

Some definitions:

- Graph H is a contraction minor of G if it can be obtained from G by only using edge contractions.
- A connected plane graph H is a partially triangulated $k \times k$-grid if $E(G_{k\times k}) \subseteq E(H) \subseteq E(G)$ holds for some triangulation G of $G_{k\times k}$.

Contraction Bidimensionality

Proposition

If a planar graph G has $G_{k\times k}$ as a minor, then it has a partially triangulated $k \times k$-grid as a contraction minor.

Proof:

Apply the contractions that obtain $G_{k\times k}$ from G but not the deletions. The result is a planar graph H with $V(H) = V(G_{k\times k})$ and $E(G_{k\times k}) \subseteq E(H)$. H can be triangulated by adding more edges. The statement follows. □

k-Planar Dominating Set is Bidimensional

Proposition

If a planar graph G contains $G_{k \times k}$ as a minor, then G has no dominating set of size less than $(k-2)^2/9$.

Proof:

By the previous proposition, G has a partially triangulated $k \times k$-grid H as a contraction minor. Let the vertices of $G_{k \times k}$ be labeled v_{ij} with $i, j \in \{1, \ldots, k\}$.
The vertices v_{ij} of H with $2 \le i \le k-1$ and $2 \le j \le k-1$ are called internal vertices of H.

k-Planar Dominating Set is Bidimensional

Proposition

If a planar graph G contains $G_{k\times k}$ as a minor, then G has no dominating set of size less than $(k-2)^2/9$.

Proof, continued:

Let S be a minimum dominating set of H. Any vertex of S dominates at most 9 internal vertices of H, hence $|S| \geq (k-1)^2/9$.
If G has a dominating set S, and G' is obtained from G by contracting $\{u, v\}$ into w, then: (1) if $u, v \notin S$, then S is a dominating set of G', and (2) if $u \in S$ or $v \in S$, then $S - u - v + w$ is a dominating set of G'. □

k-PLANAR DOMINATING SET IS BIDIMENSIONAL

Theorem

k-PLANAR DOMINATING SET can be solved in time $O(2^{O(\sqrt{k})}n^{O(1)})$.

Planar Graphs, Layers, and Grid Minors – Summary

- Many problems that (probably) do not allow FPT algorithms in general do admit FPT algorithms when restricted to planar graphs (e.g. k-INDEPENDENT SET, k-DOMINATING SET)
- 2 general methods to obtain (fast) FPT algorithms for problems of planar graphs: *layer decompositions* and *bidimensionality/grid minors*
- The layer decomposition methods tends to be faster and easier to implement.
- The bidimensionality/grid minor method is stronger, and gives easier proofs.
- Even for general graphs considering grid minors is useful for proving that an FPT algorithm exists.

Planar Graphs, Layers, and Grid Minors – Summary

- To obtain subexponential FPT algorithms for planar graphs, we need:
  (A) either a linear kernel (layers) or a bidimensionality proof (grid minors).
  (B) A dynamic programming algorithm with parameter function $2^{O(\mathrm{tw}(G))}$, and
- Bidimensionality gives fast FPT algorithms for many other graph classes that are closed under taking minors!