

Simpler Linear-Time Kernelization for Planar Dominating Set

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Abstract. We describe a linear-time algorithm that inputs a planar graph G and outputs a planar graph of size $O(k)$ and with domination number k , where k is the domination number of G , i.e., the size of a smallest dominating set in G . In the language of parameterized computation, the new algorithm is a linear-time kernelization for the NP-complete PLANAR DOMINATING SET problem that produces a kernel of linear size. Such an algorithm was previously known (van Bevern et al., these proceedings), but the new algorithm and its analysis are considerably simpler.

1 Introduction

A current trend in the area of parameterized computation is the quest for kernelization algorithms for hard computational problems. Briefly stated, a kernelization algorithm is a polynomial-time procedure that transforms an instance of the problem under consideration into an equivalent instance whose size depends only on the value of the chosen parameter. More formally, a *kernelization* for a parameterized problem $L \subseteq \Sigma^* \times \mathbb{N}$, where Σ is an alphabet and $\mathbb{N} = \{1, 2, \dots\}$, is an algorithm \mathcal{A} for which there exists a polynomial $p : \mathbb{N} \rightarrow \mathbb{N}$ and a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that, applied to an instance $I = (G, k) \in \Sigma^* \times \mathbb{N}$, \mathcal{A} computes, within $p(|I|)$ steps, an instance $I' = (G', k') \in \Sigma^* \times \mathbb{N}$ with $|I'| \leq f(k)$ and $k' \leq k$ such that $I \in L \Leftrightarrow I' \in L$. Here $|I|$ and $|I'|$ denote the number of symbols in the representation of I and I' , respectively, according to some suitable encoding scheme.

A kernelization can be valuable in the solution of a hard parameterized problem because, used as a preprocessing routine, it allows the instance size to be reduced, perhaps significantly, before an exponential-time algorithm is applied to the remaining, reduced instance. From the outset, much importance was attached to the *kernel size*, $f(k)$, and for some problems a series of results successively lowered the smallest upper bound on the size of a kernel known to be computable in polynomial time. This was the case for the problem of interest in this paper, the NP-complete PLANAR DOMINATING SET problem, formally defined as the language

$$\{(G, k) : G \text{ is an undirected planar graph, } k \in \mathbb{N} \text{ and } \text{Dom}(G) \leq k\} ,$$

where $\text{Dom}(G)$ is the *domination number* of G , i.e., the size of a smallest dominating set in G , a smallest subset D of the vertex set V of G such that every vertex in $V \setminus D$ is adjacent in G to a vertex in D . Alber et al. [1] described a first kernelization for PLANAR DOMINATING SET, whose kernel is of size at most $335k$, where k is the domination number of the input graph. The bound was subsequently lowered to $67k$ by Chen et al. [4], who also proved that no kernelization can yield a kernel of size bounded by $(2 - \epsilon)k$, for arbitrary fixed $\epsilon > 0$, unless $P = NP$. Other, more general, approaches [3,5] yield kernels of linear size for several problems defined on planar graphs, including PLANAR DOMINATING SET.

Only more recently has the *kernelization time*, $p(|I|)$, come into focus. The kernelizations for the PLANAR DOMINATING SET problem mentioned above both operate in cubic time, and so are rather slow. Traditionally, this has been largely ignored, the reasoning being that since the polynomial-time preprocessing is followed by an exponential-time computation anyway, there is little point in worrying about the degree of the polynomial. Whereas there is much truth in this argument, it is hardly advisable to be dogmatic about the issue. From the standpoint of theory, one might object that the polynomial is applied to the original instance size, n , whereas exponential means exponential in the kernel size, which may be substantially smaller than n . In practice, one can observe that the running time of a polynomial-time procedure is not always a negligible fraction of the time consumed by an exponential-time computation. In summary, it seems a worthwhile goal to try to reduce *both* the kernel size and the kernelization time.

At present, linear-time kernelizations are known for only few problems. VERTEX COVER is a case in point: The classic Buss' kernelization that yields a kernel of size $O(k^2)$ works in linear time (more powerful kernelizations for VERTEX COVER that produce kernels with $O(k)$ vertices are known [9, Section 7.4], but none of these runs in linear time). Other problems for which linear-time kernelizations were described in the literature include certain problems on planar graphs [8], CLUSTER EDITING [11] and ROOTED LEAF OUTBRANCHING [7]; as in the case of VERTEX COVER, kernels of linear size are not obtained in linear time. An exception concerns the WEIGHTED MAX LEAF problem investigated by Jansen [6], who states without giving all details that a linear-sized kernel can be obtained in linear time. For the PLANAR DOMINATING SET problem, a first kernelization algorithm that yields a kernel of linear size in linear time was described by van Bevern et al. [2]. It is based on the *region decomposition* of the earlier cubic-time kernelization algorithm of [1] and can be viewed to some extent as a more efficient implementation of that algorithm. Here we take a fresh look at the problem and develop a second linear-time kernelization for PLANAR DOMINATING SET that yields a kernel of linear size. Compared with the result of van Bevern et al., the advantage of the new result is that the algorithm and its analysis, while not trivial, are considerably simpler. On the other hand, the approach based on region decomposition may have applications to other problems defined on planar graphs, while the techniques described here are more problem-specific.

In terms of kernel size, the new algorithm, at least with its present analysis, is not competitive with the earlier algorithms mentioned above. Moreover, there is a certain trade-off between simplicity and small kernel size. For these reasons, we refrain from calculating the exact kernel size and demonstrate only as simply as we can that it is $O(k)$. Of course, if the algorithm of [4] is applied after the new algorithm, a kernel of size at most $67k$ is obtained in $O(n + k^3)$ time.

Our result is slightly more than a kernelization. As is a standard requirement for kernelizations although usually not considered part of the formal definition, our arguments imply an efficient and in fact linear-time procedure for obtaining from an arbitrary dominating set in the kernel a dominating set of the same size in the original input graph. Moreover, similarly as the algorithm of van Bevern et al. [2], our algorithm actually does not need to know the parameter k . Rather, it inputs a graph G and computes a graph G' with $\text{Dom}(G') = \text{Dom}(G)$, so that $(G, k) \in \text{PLANAR DOMINATING SET} \Leftrightarrow (G', k) \in \text{PLANAR DOMINATING SET}$ for every $k \in \mathbb{N}$. Correspondingly, a step in the kernelization that inputs a graph G and outputs another graph G' will be called *correct* if $\text{Dom}(G) = \text{Dom}(G')$.

Subsequently to the work described here, it was demonstrated that the same techniques can be employed to obtain linear-time kernelizations with linear kernel size for more general domination problems in planar graphs. These include **PLANAR ANNOTATED DOMINATING SET**, i.e., given an undirected planar graph $G = (V, E)$, a set $B \subseteq V$ and an integer $k \in \mathbb{N}$, decide whether there is a set $D \subseteq V$ with $|D| \leq k$ such that every vertex in $B \setminus D$ is adjacent in G to a vertex in D . A few of the ideas in this paper suffice to give a linear-time kernelization with linear kernel size for the **PLANAR EDGE DOMINATING SET** problem, i.e., given an undirected planar graph $G = (V, E)$ and an integer $k \in \mathbb{N}$, decide whether there is a subset $D \subseteq E$ with $|D| \leq k$ such that every edge in E shares an endpoint with an edge in D . This ongoing work will be reported elsewhere.

1.1 Overview of the New Algorithm

The starting point for the kernelization described here was the observation that if two vertices x and y in an n -vertex planar graph G with small domination number k have many joint neighbors, then most of these neighbors have no neighbors other than x and y that are not also neighbors of x or y . Then an optimal dominating set contains x or y , and most joint neighbors of x and y are without influence on the domination number and can be removed. It therefore makes sense to search for a small vertex set A such that many vertices have two neighbors in A . A good candidate for A is a smallest set P of vertices in G with total degree at least $n - |P|$.

It may not be the case that there are many vertices with two neighbors in P . Then, however, most vertices have exactly one neighbor in P , and the union Q of P with the set of vertices without neighbors in P is still small. Think of the vertices in Q as the centers of stars that include all vertices in G . If it is not optimal to dominate a star from its center, every vertex in the star must be dominated by a vertex that also dominates at least one vertex in a different star. By planarity, only few vertices dominate parts of three or more stars. As for

vertices that dominate parts of exactly two stars, at most two such dominating vertices are needed for each pair of stars, since otherwise the two stars in question could be dominated more cheaply from their centers. We cannot know which two vertices are present in an optimal solution, but for each of the two stars we can pick a single vertex that dominates a maximum number of vertices in that star and at least one vertex in the other star. By planarity, adding the vertices that were picked and those that dominate parts of more than two stars to Q still results in a small set R .

Let S be the set of centers x of stars for which the vertices in $R \setminus \{x\}$ do not have enough edges to the star of x to dominate it completely. A dominating set can be assumed to be a superset of S , and therefore every vertex in the set T of vertices at distance 1 from S can be removed unless it is needed to dominate vertices outside of $S \cup T$. This is the case only if it has two or more neighbors outside of $S \cup T$, a condition that, by planarity, holds for only few vertices in T .

If most vertices belong to stars whose centers are not in S , there may not be many vertices in T to remove. In that case, however, R is a small set such that many vertices have two neighbors in R , and we can pick $A = R$ and remove many vertices outside of R as described above. In every situation, if G contains more than Ck vertices for a suitable constant C , a constant fraction of the vertices in G can be removed in linear time, and all that remains is to repeat the process until the number of vertices no longer drops as fast as established for graphs with more than Ck vertices.

2 Preliminaries

2.1 Definitions and Notation

Throughout this subsection, let $G = (V, E)$ be an undirected graph.

When $U \subseteq V$, $G[U]$ denotes, as usual, the subgraph of G induced by U , i.e., $(U, \{\{u, v\} \in E : u, v \in U\})$.

For $u \in V$, we denote by $N_G(u)$ the set $\{v \in V : \{u, v\} \in E\}$ of neighbors of u in G and write $N_G[u]$ for $N_G(u) \cup \{u\}$. For $U \subseteq V$, $N_G(U) = \bigcup_{u \in U} N_G(u)$ and $N_G[U] = \bigcup_{u \in U} N_G[u]$. A vertex $u \in V$ *dominates* the vertices in $N_G[u]$ in G , and a vertex set $U \subseteq V$ dominates the vertices in $N_G[U]$ in G and G itself if $N_G[U] = V$.

For $A \subseteq V$ and $i \geq 0$, we denote by $N_G^i(A)$ the set $\{u \in V \setminus A : |N_G(u) \cap A| = i\}$ of vertices in $V \setminus A$ that have precisely i neighbors in A , and $N_G^{\geq i}(A) = \bigcup_{j=i}^{\infty} N_G^j(A)$.

2.2 Properties of Planar Graphs

The facts stated in Lemma 1 below are well-known. For proofs see, e.g., [10]. The remaining lemmas in this section are straightforward consequences of Lemma 1.

Lemma 1. Let $G = (V, E)$ be an undirected planar graph and take $n = |V|$ and $m = |E|$. Then

- (a) $m \leq 3n$.
- (b) If G is bipartite, then $m \leq 2n$.
- (c) No three distinct vertices in G have three common neighbors.

Lemma 2. Let $G = (V, E)$ be an undirected planar graph and let $A \subseteq V$. Then

- (a) For $i = 3, 4, \dots$, let $n_i = |N_G^i(A)|$. Then $\sum_{i=3}^{\infty} (i - 2)n_i \leq 2|A|$.
- (b) $|N_G^{\geq 3}(A)| \leq 2|A|$.
- (c) Let \mathcal{F} be a set of faces in a planar embedding of G , the boundary of each of which contains 3 or more vertices in $N_G^{\geq 2}(A)$. Then at most $6|A|$ vertices in $N_G^{\geq 2}(A)$ lie on the boundary of one or more faces in \mathcal{F} .

Proof. (a) Applying Lemma 1(b) to the subgraph of G induced by the edges with one endpoint in A and one endpoint in $N_G^{\geq 3}(A)$ yields $\sum_{i=3}^{\infty} in_i \leq 2(|A| + \sum_{i=3}^{\infty} n_i)$.

(b) Since $|N_G^{\geq 3}(A)| = \sum_{i=3}^{\infty} n_i \leq \sum_{i=3}^{\infty} (i - 2)n_i$, the claim follows from part (a).

(c) Apply part (a) to the graph obtained from G by, for each $F \in \mathcal{F}$, adding a new vertex v_F and a new edge from v_F to each vertex in $N_G^{\geq 2}(A)$ on the boundary of F . If the number of new edges is m , this shows that $m \leq 2(|A| + |\mathcal{F}|) \leq 2(|A| + m/3)$ and therefore that $m \leq 6|A|$. \square

Lemma 3. Let $G = (V, E)$ be an n -vertex undirected planar graph, let $A \subseteq V$ and assume that $\sum_{x \in A} |N_G[x]| \geq n$. For $i = 0, 1, \dots$, let $n_i = |N_G^i(A)|$. Then $n_0 \leq n_2 + 10|A|$.

Proof. The total degree in G of the vertices in A is at least $n - |A|$ and, by Lemma 1(a), at most $3|A|$ edges have both endpoints in A . Therefore

$$\sum_{i=0}^{\infty} in_i \geq n - |A| - 2 \cdot 3|A| = \sum_{i=0}^{\infty} n_i - 6|A|$$

and

$$n_0 \leq n_2 + \sum_{i=3}^{\infty} (i - 1)n_i + 6|A| \leq n_2 + 6|A| + 2 \sum_{i=3}^{\infty} (i - 2)n_i .$$

By Lemma 2(a), $\sum_{i=3}^{\infty} (i - 2)n_i \leq 2|A|$. The claim follows. \square

3 Near-Twin Reduction

A central part of our kernelization is a procedure called *near-twin reduction* that is applied to an undirected planar graph $G = (V, E)$ and parameterized by a vertex set $A \subseteq V$.

3.1 Description

With G and A as above, define the A -neighborhood of a vertex $u \in V$ as the set $N_G(u) \cap A$. For all $x, y \in A$ with $x \neq y$, call $u \in V$ *introspective* with *support* $\{x, y\}$ if $u \notin A$, the A -neighborhood of u is $\{x, y\}$, and the A -neighborhood of every vertex at distance at most 2 from u in $G[V \setminus A]$ is a nonempty subset of $\{x, y\}$. A *near-twin reduction* in G with *support* A returns the graph $G' = (V', E')$ obtained from G by the following operation: For all $\{x, y\} \subseteq A$ with $x \neq y$ such that the set $I_{\{x,y\}}$ of introspective vertices with support $\{x, y\}$ is of size at least 4, replace the vertices in $I_{\{x,y\}}$ by two new degree-2 vertices with neighbors x and y (see Fig. 1). G' is clearly planar. In Section 4, we write an application of the near-twin reduction with support A to G as a function call $NearTwin_Reduce(G, A)$ that returns the graph G' resulting from the near-twin reduction.

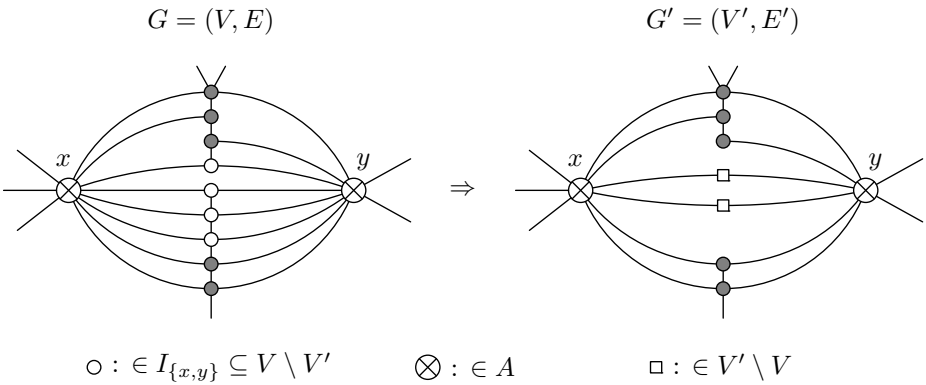


Fig. 1. A near-twin reduction with support A transforms G into G'

3.2 Correctness

Suppose that D is a dominating set in G and consider the set $D' \subseteq V'$ obtained from D as follows: For all $\{x, y\} \subseteq A$ with $x \neq y$ such that $|I_{\{x,y\}}| \geq 4$ and $D \cap N_G[I_{\{x,y\}}] \not\subseteq \{x, y\}$, replace the vertices in $D \cap N_G[I_{\{x,y\}}]$ by x and y . Note that $N_G[I_{\{x,y\}}] \cap N_G[I_{\{x',y'\}}] = \emptyset$ for all $\{x', y'\} \subseteq A$ with $x' \neq y'$ and $\{x, y\} \neq \{x', y'\}$, so that the replacement can actually be carried out as described. Since $N_G[N_G[I_{\{x,y\}}]] \subseteq N_G[\{x, y\}]$ for all $\{x, y\} \subseteq A$ with $x \neq y$, D' still dominates G , and since $D \cap N_G[I_{\{x,y\}}] \neq \emptyset$, it is easy to see that D' dominates G' . By Lemma 1(c), the vertices in a set $I_{\{x,y\}}$ with $|I_{\{x,y\}}| \geq 4$ cannot be dominated in G by any single vertex other than x or y . Therefore, if $D \cap N_G[I_{\{x,y\}}] \not\subseteq \{x, y\}$, then $|D \cap N_G[I_{\{x,y\}}]| \geq 2$. This shows that $|D'| \leq |D|$.

Suppose, conversely, that D' is a dominating set in G' and consider the set $D \subseteq V$ obtained from D' by replacing the vertices in $D' \cap (V' \setminus V)$ by their neighbors. D clearly dominates G , and $|D| \leq |D'|$ since for every vertex in $D' \cap (V' \setminus V)$ without neighbors in D' there is another vertex in $D' \cap (V' \setminus V)$ with

the same two neighbors. Altogether, we have shown that $\text{Dom}(G) = \text{Dom}(G')$, i.e., the near-twin reduction is correct.

3.3 Execution Time

The near-twin reduction with support A can be applied to G in $O(|V|)$ time as follows: First each vertex $u \in V \setminus A$ computes the set U , where U is the A -neighborhood of u if the latter is of size 1 or 2 and $U = \{\perp\}$ for a special symbol \perp otherwise. Then, in each of two successive rounds, each vertex in $V \setminus A$ broadcasts its set to all its neighbors in $V \setminus A$ and forms the new value of its own set as the union of the old value and the sets received, except that every union that contains \perp or is of size 3 or more is replaced by $\{\perp\}$. A vertex is introspective exactly if its stored set is of size 2 throughout this process. The introspective vertices can be sorted by their supports in $O(|V|)$ time with radix sort, after which it is a simple matter to construct G' .

3.4 Reduction Progress

Lemma 4. *Let $G = (V, E)$ be an undirected planar graph, let $A \subseteq V$ and let $G' = (V', E')$ be the graph obtained by carrying out a near-twin reduction in G with support A . Then $|N_{G'}^2(A)| = O(|A| + \text{Dom}(G))$.*

Proof. We first show that in G' , no four introspective vertices have a common support $\{x, y\}$. Otherwise some vertex $u \in (V \cap V') \setminus A$ with A -neighborhood $\{x, y\}$ would be introspective in G' , but not in G . This can happen only if the near-twin reduction in G removes a vertex v at distance at most 2 from u in $G[V \setminus A]$ whose A -neighborhood is not $\{x, y\}$. But this is impossible since the A -neighborhood of u is $\{x, y\}$ and u is within distance 2 of v in $G[V \setminus A]$ —the presence of u would prevent v from being introspective, and therefore from being removed.

Since clearly there is a planar graph on the vertex set A that contains an edge $\{x, y\}$ if there is an introspective vertex with support $\{x, y\}$, Lemma 1(a) shows that the number of introspective vertices in $U = N_{G'}^2(A)$ is bounded by $3 \cdot 3|A|$. To bound the number of the remaining vertices in U , fix a planar embedding of G' and its restriction ϕ to the subgraph of G' induced by the set of edges with one endpoint in A and one endpoint in U . Denote by \mathcal{F} the set of faces of ϕ . We can clearly assume that $|U| \geq 3$. Then $\mathcal{F} = \mathcal{F}_2 \cup \mathcal{F}_{\geq 3}$, where \mathcal{F}_2 and $\mathcal{F}_{\geq 3}$ are the sets of faces in \mathcal{F} whose boundary contains exactly 2 and ≥ 3 distinct vertices in U , respectively. Moreover, every face in \mathcal{F} whose boundary contains three distinct vertices in A belongs to $\mathcal{F}_{\geq 3}$.

Let $x, y \in A$ be distinct and let u be a vertex in U whose A -neighborhood is $\{x, y\}$. If u is not introspective, there is a path π in $G[V \setminus A]$ of length at most 2 from u to a vertex v whose A -neighborhood is not a nonempty subset of $\{x, y\}$. Choose π as a shortest such path. Then one of the following holds (see Fig. 2):

- $v \in U$, and every face in \mathcal{F} whose boundary contains both v and the last vertex in U that precedes v on π belongs to $\mathcal{F}_{\geq 3}$, since its boundary includes the A -neighborhoods of both u and v .

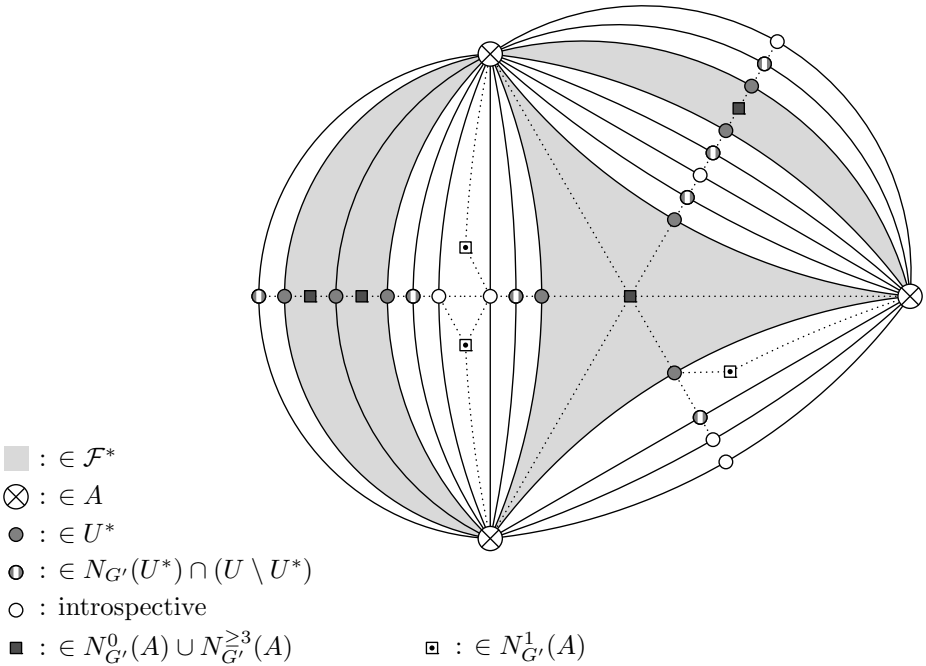


Fig. 2. Vertices in U sufficiently far from the faces in \mathcal{F}^* are introspective

- $v \in N_{G'}^1(A)$, and the face in \mathcal{F} whose interior contains v belongs to $\mathcal{F}_{\geq 3}$, since its boundary includes the A -neighborhoods of both u and v .
- $v \in N_{G'}^0(A) \cup N_{G'}^{\geq 3}(A)$.

Let \mathcal{F}^* be the union of $\mathcal{F}_{\geq 3}$ with the set of faces in \mathcal{F}_2 whose interior contains a vertex in $N_{G'}^0(A) \cup N_{G'}^{\geq 3}(A)$ and let U^* be the set of vertices in U on the boundary of a face in \mathcal{F}^* . The considerations above show that every vertex in U that is not introspective is at distance at most 1 in G' from a vertex in U^* .

Let F be a face in \mathcal{F}_2 whose interior contains a vertex $v \in N_{G'}^0(A)$. A vertex that dominates v must lie in the interior of F or be one of its two boundary vertices in U , each of which can dominate vertices in at most one face other than F . This shows the number of vertices in U on the boundaries of faces in \mathcal{F}_2 whose interiors contain a vertex in $N_{G'}^0(A)$ to be at most $3\text{Dom}(G') = 3\text{Dom}(G)$. By Lemma 2(b), the number of vertices in U on the boundaries of faces in \mathcal{F}_2 whose interiors contain a vertex in $N_{G'}^{\geq 3}(A)$ is bounded by $2 \cdot 2|A|$. And, finally, Lemma 2(c) shows the number of vertices in U on the boundaries of faces in $\mathcal{F}_{\geq 3}$ to be at most $6|A|$. In summary, $|U^*| \leq 10|A| + 3\text{Dom}(G)$. Each vertex in U^* lies on the boundary of at most one face in $\mathcal{F} \setminus \mathcal{F}^* \subseteq \mathcal{F}_2$ and therefore has at most one neighbor in $U \setminus U^*$. Altogether, U contains at most $9|A|$ introspective vertices and at most $2(10|A| + 3\text{Dom}(G))$ vertices that are not introspective. \square

4 Shrinking Iterations

4.1 Description

The main part of the kernelization is the *shrinking iteration* detailed below. It takes as input an undirected planar graph $G = (V, E)$, goes through three phases that successively transform G into G_1 , G_2 and G_3 , and returns G_3 . Take $n = |V|$ and, for $i = 1, 2, 3$, write $G_i = (V_i, E_i)$.

The first two phases are simply near-twin reductions with different supports. The support used in Phase 1 is a smallest set $P \subseteq V$ with $\sum_{x \in P} |N_G[x]| \geq n$. To describe the support used in Phase 2, we need some additional notation.

First, let Q be the union of P with the set of vertices in V_1 without neighbors in P . Q dominates G_1 , so we can form a partition $\{Star(x) : x \in Q\}$ of V_1 with $x \in Star(x) \subseteq N_{G_1}[x]$ for all $x \in Q$. For all $x \in Q$, we call $Star(x)$ a *star* and x its *center*. For all $u \in V_1$, let $L(u) = \{x \in Q : N_{G_1}[u] \cap Star(x) \neq \emptyset\}$ be the set of centers of stars represented in $N_{G_1}[u]$. Moreover, for all $x, y \in Q$ with $x \neq y$, let $w_y(x)$ be the supremum, over all $u \in V_1 \setminus \{x\}$ with $L(u) = \{x, y\}$, of $|N_{G_1}[u] \cap Star(x)|$ (equal to $-\infty$ if there are no such u) and let $g_y(x)$ be a vertex u that realizes the supremum (undefined if $w_y(x) = -\infty$). Phase 2 is a near-twin reduction with support

$$R = Q \cup \{g_y(x) : x, y \in Q, x \neq y \text{ and } w_y(x) > 0\} \cup \{u \in V_1 : |L(u)| \geq 3\} .$$

Let $w(x) = \sum_{y \in R \setminus \{x\}} |N_{G_1}[y] \cap Star(x)|$ for all $x \in Q$, $S = \{x \in Q : w(x) < |Star(x)|\}$ and $T = N_{G_2}(S) \setminus S$. Phase 3 consists in removing every vertex in T with at most one neighbor outside of $S \cup T$ and, for each $x \in S$, adding a new degree-1 vertex with neighbor x . A nonobvious point is that since the definition of S refers to G_1 , S is best computed before the near-twin reduction in Phase 2.

The three phases are summarized below in pseudo-code.

// Phase 1

$P :=$ a smallest subset of V with $\sum_{x \in P} |N_G[x]| \geq n$;

$G_1 := (V_1, E_1) := \text{NearTwin_Reduce}(G, P)$;

// Phase 2

$Q := P \cup N_{G_1}^0(P)$;

$\{Star(x) : x \in Q\} :=$ a partition of V_1 with $x \in Star(x) \subseteq N_{G_1}[x]$ for all $x \in Q$;

$R := Q \cup \{g_y(x) : x, y \in Q, x \neq y \text{ and } w_y(x) > 0\} \cup \{u \in V_1 : |L(u)| \geq 3\}$,

where $L(u) = \{x \in Q : N_{G_1}[u] \cap Star(x) \neq \emptyset\}$ for all $u \in V_1$,

$w_y(x) = \sup\{|N_{G_1}[u] \cap Star(x)| : u \in V_1 \setminus \{x\} \text{ and } L(u) = \{x, y\}\}$

for all $x, y \in Q$ with $x \neq y$ and

$g_y(x) =$ some $u \in V_1 \setminus \{x\}$ with $L(u) = \{x, y\}$

and $|N_{G_1}(u) \cap Star(x)| = w_y(x)$

for all $x, y \in Q$ with $x \neq y$ and $w_y(x) > 0$;

$S := \{x \in Q : w(x) < |Star(x)|\}$,

where $w(x) = \sum_{y \in R \setminus \{x\}} |N_{G_1}[y] \cap Star(x)|$ for all $x \in Q$;

$G_2 := (V_2, E_2) := \text{NearTwin_Reduce}(G_1, R)$;

// Phase 3

$T := N_{G_2}(S) \setminus S;$

$V'_3 := V_2 \setminus \{u \in T : |N_{G_2}(u) \setminus (S \cup T)| \leq 1\};$

$G_3 := (V_3, E_3) :=$ the graph obtained from $G_2[V'_3]$ by adding,

for each $x \in S$, a new degree-1 vertex with neighbor x ;

return G_3 ;

4.2 Correctness

It was proved already in Section 3.2 that near-twin reductions preserve the domination number. Demonstrating the correctness of a shrinking iteration therefore boils down to showing that $Dom(G_2) = Dom(G_3)$.

Let D_3 be a dominating set in G_3 . Since $N_{G_3}(V_3 \setminus V_2) = S$ and $V_2 \setminus V_3 \subseteq N_{G_2}(S)$, $D_2 = (D_3 \cap V_2) \cup S$ is a dominating set in G_2 . And since every vertex in S has a degree-1 neighbor in G_3 that does not belong to V_2 , $|D_2| \leq |D_3|$.

Conversely, let D_1 be a minimum dominating set in G_1 (not in G_2), chosen to maximize $|D_1 \cap Q|$ among all such sets. D_1 does not contain any vertex $u \in V_1 \setminus Q$ with $|L(u)| = 1$, since every such vertex could be replaced in D_1 by the center of its star to obtain a minimum dominating set with one more element in Q . Likewise, for all $x, y \in Q$ with $x \neq y$, D_1 contains at most one vertex $u \in V_1 \setminus \{x\}$ with $L(u) = \{x, y\}$, since two such vertices could be replaced by x and y in D_1 to obtain a minimum dominating set with at least one more element in Q . R contains all $u \in V_1$ with $|L(u)| \geq 3$ and, for all $x, y \in Q$ with $x \neq y$ such that $L(u) = \{x, y\}$ for some $u \in V_1 \setminus \{x\}$, one such vertex u for which $|N_{G_1}[u] \cap Star(x)|$ is maximal. Because of this, it is not difficult to see that for all $x \in Q$, $w(x) = \sum_{y \in R \setminus \{x\}} |N_{G_1}[y] \cap Star(x)|$ is an upper bound on the number of vertices in $Star(x)$ that are dominated by $D_1 \setminus \{x\}$. If $w(x) < |Star(x)|$ for some $x \in Q$, therefore, $x \in D_1$. Hence $S \subseteq D_1$.

The argument in Section 3.2 that shows that $Dom(D_2) \leq Dom(D_1)$ also proves that there is a minimum dominating set D_2 in G_2 with $D_1 \cap R \subseteq D_2 \cap R$ and therefore $S \subseteq D_2$. Let D_2 be such a set and obtain $D_3 \subseteq V_3$ from D_2 by replacing each vertex $u \in D_2 \setminus V_3$ by the vertices in $N_{G_2}(u) \setminus (S \cup T)$. In G_3 , D_3 dominates the vertices in $V_3 \cap (S \cup T)$ (because $S \subseteq D_3$), the vertices in $V_2 \setminus (S \cup T)$ (by the way D_3 is obtained from D_2), and the vertices in $V_3 \setminus V_2$ (again because $S \subseteq D_3$). Therefore D_3 is a dominating set in G_3 . Since $|N_{G_2}(u) \setminus (S \cup T)| \leq 1$ for each $u \in V_2 \setminus V_3$, $|D_3| \leq |D_2|$. Thus $Dom(G_2) = Dom(G_3)$.

4.3 Execution Time

It was proved already in Section 3.3 that a near-twin reduction can be executed in linear time. To compute P , sort the vertices by their degrees and pick a suitable suffix of the sorted sequence. To compute Q , let each vertex determine membership in P for itself and its neighbors to know whether it should enter Q . The computation of T from S and of V'_3 from S and T are similar, and the construction of G_3 from G_2 , V'_3 and S is easy.

In order to compute the partition $\{Star(x) : x \in Q\}$, let each vertex in $V_1 \setminus Q$ choose a neighbor $x \in Q$ and mark itself as belonging to $Star(x)$. Subsequently each vertex $u \in V_1$ can determine the number $|L(u)|$ of stars represented in $N_{G_1}[u]$ and enter R if $|L(u)| \geq 3$. If $|L(u)| = 2$, instead let u generate the tuples (x, y, r_x, u) (unless $u = x$) and (y, x, r_y, u) (unless $u = y$), where $L(u) = \{x, y\}$, $r_x = |N_{G_1}[u] \cap Star(x)|$ and $r_y = |N_{G_1}[u] \cap Star(y)|$. Sorting the tuples generated in this way lexicographically with radix sort allows us to compute the final constituent $\{g_y(x) : x, y \in Q, x \neq y \text{ and } w_y(x) > 0\}$ of R in $O(n)$ time: For all $x, y \in Q$ with $x \neq y$ and $w_y(x) > 0$, $g_y(x)$ is the fourth component of the last tuple in the sorted sequence with first component x and second component y . The computation of S , finally, is made easy by the alternative characterization $w(x) = \sum_{u \in Star(x)} |N_{G_1}[u] \cap (R \setminus \{x\})|$, for all $x \in Q$, which shows that $w(x)$ can be obtained for all $x \in Q$ by summing over the vertices in $Star(x)$ a quantity that is computable for all vertices in V_1 in $O(n)$ time. The application of a shrinking iteration to an n -vertex graph therefore takes $O(n)$ time.

4.4 Reduction Progress

Take $k = Dom(G)$ and note that $|P| \leq k$. Let $z = |V| - |V_1| = |N_G^2(P)| - |N_{G_1}^2(P)| \geq 0$ be the decrease in the number of vertices achieved by Phase 1. By Lemma 4, $|N_{G_1}^2(P)| = O(k)$, so $|N_G^2(P)| = z + O(k)$. Lemma 3 now shows that $|N_G^0(P)| \leq |N_G^2(P)| + O(k) = z + O(k)$. And then $|Q| = |P| + |N_G^0(P)| = z + O(k)$.

Let $R_{\geq 3} = \{u \in V_1 \setminus Q : |L(u)| \geq 3\}$. We want to show that $|R_{\geq 3}| = O(|Q|)$, which, by Lemma 1(a), will imply that $|R| \leq |Q| + 6|Q| + |R_{\geq 3}| = O(z + k)$. To this end, recall that for $t \in \mathbb{N}$, a t -coloring of an undirected graph $H = (V_H, E_H)$ is a mapping $h : V_H \rightarrow \{1, \dots, t\}$ such that for every $\{u, v\} \in E_H$, $h(u) \neq h(v)$. Lemma 1(a) can easily be used to show that every undirected planar graph has a 6-coloring, and the famous four-color theorem (see, e.g., [10]) states that every undirected planar graph in fact has a 4-coloring. Therefore let h be a t -coloring of G_1 for some $t \leq 6$ and fix $j \in \{1, \dots, t\}$. Consider the graph obtained from G_1 by merging each vertex $u \in V_1 \setminus Q$ with $h(u) \neq j$ into the center of its star. In the resulting graph every vertex $u \in R_{\geq 3}$ with $h(u) = j$ has at least 3 neighbors in Q . Therefore, by Lemma 2(b), $|\{u \in R_{\geq 3} : h(u) = j\}| \leq 2|Q|$, and altogether $|R_{\geq 3}| \leq 2t|Q| = O(|Q|)$.

Let $W = \sum_{x \in Q} w(x)$. Intuitively, W is (approximately) the number of vertices that, by virtue of having a neighbor in R other than the center of their star, can be “saved” from entering T . Under the assumption that z is small, so that Phase 1 removes only few vertices, the following holds: If W is small, T is large, and Phase 3 succeeds in removing a large number of vertices in T . On the other hand, if W is large, then many vertices have two neighbors in R , and the near-twin reduction with support R in Phase 2 removes many vertices. The rest of this section serves to formalize and prove these claims. First note that

$$\begin{aligned} |N_{G_1}[S]| &\geq \sum_{x \in Q: w(x) < |Star(x)|} |Star(x)| \geq \sum_{x \in Q: w(x) < |Star(x)|} (|Star(x)| - w(x)) \\ &\geq \sum_{x \in Q} (|Star(x)| - w(x)) = |V_1| - W . \end{aligned}$$

Thus at most W vertices in G_1 do not belong to $N_{G_1}[S]$. This implies that at most W vertices in G_2 do not belong to $N_{G_2}[S] = S \cup T$. And this in turn means that $|V_2 \setminus T| \leq W + |S|$. A vertex in T is included in V'_3 only if it has at least two neighbors outside of T in addition to the center of its star. By Lemma 2(b), therefore, $|V'_3| \leq |V_2 \setminus T| + 2|V_2 \setminus T| \leq 3(W + |S|)$ and $|V_3| = |V'_3| + |S| \leq 3W + O(z + k)$. This concludes the analysis for small W . Take $U = V_1 \setminus R$. The analysis for large W depends on the following characterization of W .

$$\begin{aligned} W &= \sum_{x \in Q} \sum_{y \in R \setminus \{x\}} |N_{G_1}[y] \cap Star(x)| = \sum_{y \in R} \sum_{x \in Q \setminus \{y\}} |N_{G_1}[y] \cap Star(x)| \\ &= \sum_{y \in R} |N_{G_1}[y]| - \sum_{y \in Q} |Star(y)| = \sum_{y \in R} |N_{G_1}(y)| - |U| . \end{aligned}$$

Let H be the bipartite subgraph of G_1 induced by the set of edges with one endpoint in R and one endpoint in U . The total degree $\sum_{y \in R} |N_{G_1}(y)|$ in G_1 of the vertices in R overcounts the number of edges in H by twice the number of edges between vertices in R , i.e., according to Lemma 1(a), by at most $6|R|$. Therefore H has at least $|U| + W - 6|R|$ edges. Every vertex in U has at least one incident edge in H , namely to the center of its star. With $n_i = |N_{G_1}^i(R)|$ for $i = 1, 2, \dots$, this means that $\sum_{i=1}^{\infty} (i - 1)n_i \geq W - 6|R|$. But then, by Lemma 2(a),

$$\sum_{i=2}^{\infty} n_i \geq W - 6|R| - \sum_{i=3}^{\infty} (i - 2)n_i \geq W - 6|R| - 2|R| = W - O(z + k) .$$

After Phase 2, by Lemmas 4 and 2(b), the number of vertices with two or more neighbors in R is $O(z + k)$. Phase 2 therefore reduces the number of vertices with two or more neighbors in R from $W - O(z + k)$ to $O(z + k)$, i.e., by $W - O(z + k)$.

In summary, for a certain constant $c > 0$, Phase 1 and Phase 2 reduce the number of vertices by z and by at least $\max\{W - c(z + k), 0\}$, respectively, while the number of vertices left after Phase 3 is bounded by $3W + c(z + k)$. The complete shrinking iteration therefore reduces the number of vertices by at least

$$\max\{z, W - c(z + k), n - 3W - c(z + k)\} .$$

A simple case analysis shows that if $n \geq 16ck$, then one of the three arguments of the maximum above is $\Omega(n)$: Without loss of generality, $z \leq n/(16c)$ and therefore $c(z + k) \leq n/16 + n/16 = n/8$. If $W \geq n/4$, the second term is at least $n/4 - n/8 = n/8$, and if not, the third term is at least $n - 3n/4 - n/8 = n/8$.

A shrinking iteration that starts with at least $16ck$ vertices therefore reduces the number of vertices by at least a constant factor.

5 The Complete Algorithm

The complete kernelization inputs an undirected n -vertex planar graph G and consists in applying repeated shrinking iterations to G until the number of vertices no longer drops at least at the rate established in the analysis in the previous section. Since our upper bound on the running time of successive iterations forms a geometric series, the total running time is $O(n)$. Every iteration preserves the domination number, so the final graph has domination number $Dom(G)$, and it is planar and contains $O(Dom(G))$ vertices.

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