Definition 1 (Hypergraph).

Hypergraph is a pair $G = (V, E)$ where V is set of elements called *vertices* or *nodes* and $E \subseteq \{e \subset V \mid |e| \geq 2\}$ is set of elements called *hyperedges* or *edges*.

Definition 2 (Hitting Set).

Let (S, C) be a hypergraph. A set $H \subseteq S$ is a *hitting set* of (S, C) if and only if H has nonempty intersection with every element of C.
In case the sizes of the elements of C are bounded from above by some fixed d, the problem is called *d-hitting set*.

Definition 3 (Hitting Set Problem).

Given hypergraph (S, C) and parameter k, decide if there is a hitting set H of (S, C) such that $|H| \leq k$.

Definition 4 (Crown decomposition).

Let $G = (V, E)$ be a graph, then a (H, I, M) is crown decomposition of G if and only if the following holds:

- I is a non-empty independent set of G.
- H is $N_G(I)$
- M is H-saturating matching of the (bipartite) graph $G_M = (V_M, E_M)$ where $V_M = I \cup H$ and $E = \{\{u, v\} \mid u \in I \wedge v \in H\}$.

Lemma 5.

Graph $G = (V, E)$ has crown decomposition if and only if there is a non-empty independent set I of G such that $|I| \geq |N_G(I)|$.

Proof.

$\Rightarrow$: Immediate.

$\Leftarrow$: Using Hall's Marriage Theorem. ∎

There is an algorithm that for given graph G and its independent set I such that $|I| \geq |N_G(I)|$ finds, in polynomial time, crown decomposition (H', I', M) of G such that $H' \subseteq N_G(I)$, $I' \subseteq I$. We will refer to this algorithm as `Construct-Crown`.

Definition 6 (Crown decomposition for hypergraphs).

Let $G = (V, E)$ be a hypergraph, then (H, I, M) is crown decomposition of G if and only if the following holds:

- I is an independent set of G.
- $H = \{J \subset V \mid \exists x \in I \,.\, J \cup \{x\} \in C\}$.
- M is matching of the (bipartite) graph $G_M = (V_M, G_M)$ where
  - $V_M = I \cup H$
  - $E_M = \{(x, J) \mid x \in I \wedge J \in H \wedge (\{x\} \cup J) \in C\}$.
- Every element of H is matched under M.

Lemma 7.

Let (S, C, k) be an instance of 3HS and let (H, I, M) be a crown of (S, C). Then (S, C, k) is equivalent to the instance $(S - I, (C - E(I)) \cup H, k)$.

Proof.

Let A be a solution for 3HS instance (S, C, k). Let $B \subseteq H$ be a set that contains exactly the edges of H with nonempty intersection with A. Then for every pair $\{x, y\}$ from $(H - B)$ there is an edge $\{\{x, y\}, z\} \in M$ where $z \in A \cap I$ is used to cover the edge $\{x, y, z\} \in C$. Replacing z by any element of $\{x, y\}$ produces another hitting set A' such that $|A'| \leq |A|$. ∎

Lemma 8 (High-Degree rule 3HS).

If $x \in S$ has more than k edges whose pairwise intersection is $\{x\}$, then:

- *Add x to any potential solution for (S, C, k).*
- *Remove elements of $E(x)$ from the instance and decrease k by 1.*

Proof.

Soundness of the rule is obvious. ■

To verify that the condition holds for some vertex x is to find a matching M of the simple graph obtained by deleting x from $E(x)$ such that $|M| > k$. This can be done in polynomial time.
If the high degree rule does not apply to any vertex of 3HS instance (S, C, k), we shall say that (S, C, k) is **reduced**.

Definition 9 (Weakly related edges).

Let (S, C, k) be a 3HS instance. Two distinct elements e_1, e_2 of C are *weakly related edges* if and only if $|e_1 \cap e_2| \leq 1$. Set $W \subseteq C$ is maximal collection of weakly related edges if every edge of C is either in W or there is some $e' \in W$ such that $|e \cap e'| \geq 2$.

Lemma 10.

Let (S, C, k) be a reduced yes instance of 3HS and let W be a collection of weakly related edges of (S, C), then $|W| \leq k^2$.

Proof.

There is some set $A \subseteq S$ such that it covers all the edges in W and $|A| \leq k$. Since the instance is reduced, by the high-degree rule, every vertex of A belongs to at most k edges of W. ■

We can construct *maximal* set W of weakly related edges by a simple greedy approach, starting with (any) singleton set and filing it with elements of C as long as there is one that satisfies the condition.
We will construct W while reducing the instance (using the high-degree rule) at the same time.

Lemma 11 (Crown properties).

Let (S, C, k) *be reduced yes instance of 3HS and let* W *be a maximal set of weakly related edges of* (S, C)*,* $I = \{x \in S \mid x \notin V(W)\}$ *and* $H = \{\{x, y\} \mid \exists e \in W \,.\, \{x, y\} \subset e\}$*. WLOG assume that* W *contains all edges of* C *of size less than 3.*

1. I *is an independent set.*
2. $|S - I| \leq 2k^2 + k$*.*
3. *Every edge* e *of size three that contains element* $x \in I$ *consists of* x *and a pair from* H*.*
4. $|H| \leq 3k^2$*.*

Crown properties.

1. Assume that there is an edge e containing two elements of I, such an edge could be added to W (by definition of I), which contradicts the maximality of W.
2. The vertices from $S - I$ are exactly the vertices that are contained in some edge from W. We know that there is some solution A (of size at most k) that covers all elements of W. Since the instance is reduced, every element x of A belongs to at most k elements of W. This gives us at most $2k$ vertices of $V(W)$ per one vertex of A. Thus $S - I$ contains at most $2k^2 + k$ vertices.

■

Crown properties.

3. Assume that $e = \{x, y, z\}$ such that $\{x, y\} \notin H$, thus intersection of e with any element of W would contain at most one element, which would contradict the maximality of W.
4. Each edge of size (at most) three gives rise to (at most) three (unique) elements of H.

■

Consider the simple bipartite graph $G_{I,H} = (H \cup I, E_I)$ where edges of E_I are simple edges connecting an element $x \in I$ to an element $\{y, z\} \in H$ if $\{x, y, z\} \in C$.
By Lemma 5, we can apply `Construct-Crown` to find crown (H', I', M) for $G_{I,H}$. It is easy to see that this crown is also crown for the original hypergraph.

Algorithm 1 Kernelization of 3HS

while C contains singleton set $\{x\}$ **do**
  Add x to A
  Delete all edges that contain x from C
  Decrement k
end while
Reduce the instance and construct W of the reduced instance
if $|W| > k^2$ **then**
  Return **false**
end if
if $|I| \geq |H|$ **then**
  $(H', I', M) \leftarrow \texttt{Construct-Crown}(G_{I,H})$
  $S \leftarrow S - I'$
  $C \leftarrow (C - E(I')) \cup H'$
end if
Return the (possibly) new instance.

Theorem 12.

There is a polynomial time algorithm that, for an arbitrary input instance (S, C, k) of 3HS, either determines that the instance does not have a solution or computes a kernel whose order is bounded from above by $5k^2 + k$.

Proof.

In case $|I| < |H|$ we have $|S| = |S - I| + |I| \leq 2k^2 + k + 3k^2 = 5k^2 + k$.
In case $|I| \geq |H|$:

1. $|I - I'| \leq |H - H'|$.
2. No edges of W were removed from the instance and W is still maximal.

■

Definition 13 (Subedge).

Let $G = (S, C)$ be a hypergraph. Set $e \subseteq S$ is subedge of G if and only if $(\exists e' \in C \,.\, e \subseteq e')$. We say that subedge e is j-subedge if and only if $|e| = j$.

Lemma 14 (High-degree rule for dHS).

If $e \subset S$ is a $(d-2)$ subedge and e is the pairwise intersection of more than k edges, then delete all elements of C that contain e as a subset and add e to C.

Proof.

Soundness of the rule is obvious. ■

Definition 15 (Reduced instance).

If the high-degree rule can not be applied to a dHS instance (S, C, k), we shall say that (S, C, k) is **reduced**.

Verifying that $(d-2)$-subedge satisfies the condition of the high-degree rule is equivalent to solving the maximum matching problem in the (simple) graph given by edges $E_e = \{e' \mid e \cup e' \in C\}$. Thus it can be verified in polynomial time.

Definition 16 (Weakly related edges).

Two edges are weakly related if their intersection is not larger than $(d-2)$ and if neither one of them is a subset of the other.

Lemma 17.

Let A be a solution of reduced instance (S, C, k) of dHS, and let W be maximal set of weakly related edges. Denote by W_e the set of edges of W that properly contain e. If e is $(d-2)$-subedge then, by the high-degree rule, $|W_e| \leq k$.

Algorithm 2 High-occurence rule

```
for i = d − 2 → 1 do
    for each i-subedge e of W do
        if |W_e| > k^(d−1−i) then
            Delete (from C and W) all edges containing e
            Add e to W (and thus C)
        end if
    end for
end for
```

Soundness of the high-occurence rule.

By induction on i:

1. Base follows directly from high-degree rule (since the instance is reduced).
2. Assuming that i-th iteration is correct, we will show that every $(i-1)$-subsedge e such that $|W_e| > k^{d-i}$ has non-empty intersection with any solution A.

■

Lemma 18.

Let (S, C, k) be a reduced yes instance of dHS and let W be a maximal set of weakly related edges given by the high-occurrence rule. Then $|W| \leq k^{d-1}$.

Proof.

Every 1-subedge occurs in at most k^{d-2} edges. The elements of some solution of size (at most) k form (at most) k singleton subedges. Every element of W has to be covered by one of these subedges, thus the number of edges in W can not be greater than k^{d-1}. ∎

Lemma 19 (Crown properties).

Let (S, C, k) be our yes instance of dHS after applying the high-degree and high-occurrence rule and let W be the resulting set of weakly related edges of the instance. Let $I = \{x \in S \mid x \notin V(W)\}$ and H is a set of $(d-1)$-subedges of W. Then:

1. *I is an independent set.*
2. $|S - I| \leq (d-1)k^{d-1} + k$.
3. *Every edge e of size d that contains element $x \in I$ consists of x and $(d-1)$-subedge from H.*
4. $|H| \leq dk^{d-1}$.

Proof.

Analogous to the *3HS* case. ■

As was the case for *3HS*, we can find the crown of $G_{H,I}$ using the `Construct-Crown` algorithm. The resulting algorithm for *dHS* kernelization follows the same pattern as the one for *3HS*.