Bidimensionality: An Overview

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Outline

Bidimensionality works with:

- $r \times r$ grids
- bounded treewidth (**tw**)
- H-minor-free graphs

Bidimensionality theory includes:

- graph structural results
- framework for FPT algorithms
- polynomial-time approximation schemes (PTAS)

Bounded Treewidth

- w-tree-decomposition is FPT
- allows nice dynamic FPT algorithms
- many problems are FPT on bounded **tw**
- $r \times r$ grid $=$ typical graph of $\mathbf{tw}(r)$
  (cops and robber game)

Courcelle's (meta)theorem

Let φ be a MSOL formula and G be a graph with $\mathbf{tw}(G) \leq w$. Then in time $f(|\varphi|, w)\mathcal{O}(|V(G)|)$ it can be decided whether G satisfies φ.

Graph Minors

1. deleting vertices
2. deleting edges
3. contracting edges

Forbidden minor characterization

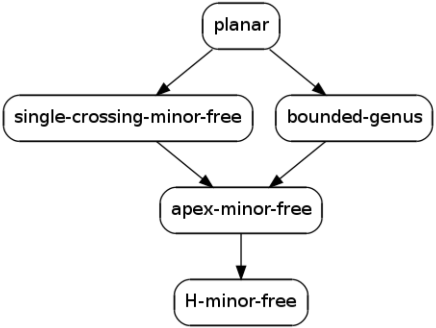
Each graph class closed on minors has a finite set of forbidden minors ($\Leftarrow$ Robertson–Seymour theorem).

- non-constructive (some forbidden minors unknown)
- H-minor problem $\mathcal{O}(n^3)$ for fixed H (superexponential w.r.t $|H|$)

-Minor-Free Graphs

- trees (triangle)
- planar graphs (K_5 and $K_{3,3}$)
- graphs embeddable on a fixed topological surface
- $\textbf{tw}(G) \leq 3$
- many are easier for FPT algorithms

Minor-Free Graphs Classification



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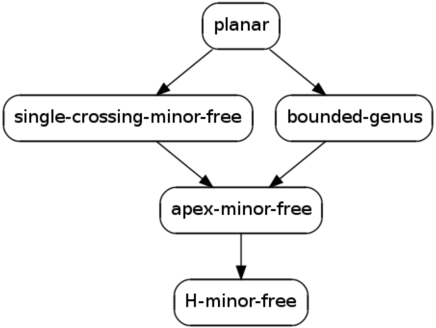

Bounded Genus Graph

Embeddable on a surface with bounded Euler genus (i.e. bounded number of *handles* and *cross-caps*).

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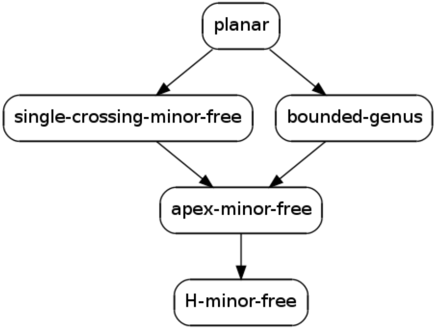
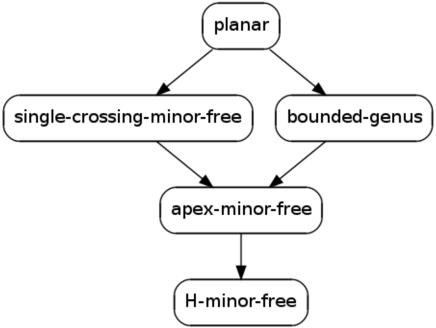


Single-crossing Graph

Can be drawn on a plane with at most two edges crossing.

Minor-Free Graphs Classification



Apex Graph

Planar after removing a vertex (apex vertex).

Large Grid Minors

- every planar graph of **tw** $\geq 6k - 5$ has a $G_{k \times k}$ minor
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Theorem

For any fixed graph H, every H-minor-free graph of **tw** $= k$ has a $G_{\Omega(k)\times\Omega(k)}$ minor.

Proof Outline

1. H-minor-free graphs can be decomposed into a clicque sum of almost embeddable graphs (R.S.)
2. clicque sum does not increase **tw** $\rightarrow$ component with greatest **tw**
3. almost embeddable reduced to embeddable
4. bounded genus has large grid (prior results – extended from planar)

Clique Sum

Let G_1 and G_2 contain a k-clique W. $G_1 \oplus 1 G_2$:

1. delete some or no edges from W
2. attach $G_1 \setminus W$ to corresponding vertices
3. attach $G_2 \setminus W$ similarly

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$\mathbf{tw}(G_1 \oplus G_2) \leq \max\{\mathbf{tw}(G_1), \mathbf{tw}(G_2)\}$

1. $\mathbf{tw}(G_1) \geq \mathbf{tw}(W), \mathbf{tw}(G_2) \geq \mathbf{tw}(W)$
2. W is in one bag in tree decompositions of G_1, G_2
3. we can get tree decomposition of $G_1 \oplus G_2$ by joining them by a bag containing only W

Bidimensional Problems

Minor Bidimensionality

A parameter P is $g(r)$-minor-BiD if it:

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- $\Theta(r^2)$-minor-BiD – size of: vertex cover, feedback vertex set, ...
- not minor-BiD: dominating set, Hamiltonian path (removing edges)

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Grid-like Graphs

planar partially triangulated $G_{r\times r}$

bounded-genus above $+ \operatorname{genus}(G)$ additional edges

apex-minor-free $G_{r\times r}$ + additional edges so that each vertex is incident to c noboundary vertex (depending on forbidden minor)

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Dominating set ($\Theta(r^2)$), Hamiltonian path ($\Theta(r^2)$), ...

FPA Schema

Let P be $g(r)$-minor bidimensional and G be H-minor-free for some H. Is $P(G) \geq k$?

1. G has a $G_{\mathbf{tw}(G)\times\mathbf{tw}(G)}$, so $P(G) \geq g(\mathbf{tw}(G))$
2. if $g(\mathbf{tw}(G)) \geq k$, return YES
3. otherwise, $\mathbf{tw}(G) \leq k \implies$ use dynamic programming (or Courcelle's theorem)

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Can we get subexponential FPTA (e.g. $\mathcal{O}(2^{\sqrt{k}})$)?

Parameter-Treewidth Bounds

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Let P be $g(r)$-BiD for the graph G. Then $\mathbf{tw}(G) = \mathcal{O}\left(g^{-1}(P(G))\right)$.

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Proof (minor-BiD): G has a $G_{\Omega(r)\times\Omega(r)}$ minor R. $P(R) \leq g(r) \geq g(\Omega(\mathbf{tw}(G)))$. Since P does not increase when taking minors, $P(G) \geq P(R)$, i.e. $P(G) \geq g(\Omega(\mathbf{tw}(G)))$ and $\mathbf{tw}(G) = \mathcal{O}\left(g^{-1}(P(G))\right)$.

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If $g(r) = \Theta(r^2)$ then $\mathbf{tw}(G) = \mathcal{O}\left(\sqrt{P(G)}\right)$.

Subexponantial FPTAs

Theorem

Assume P is $g(r)$-BiD and can be computed in $h(w)n^{\mathcal{O}(1)}$. Then there is an algorithm that computes $P(G)$ for a graph class corresponding to P which has the complexity of $\left(h(\mathcal{O}(g^{-1}(k))) + 2^{\mathcal{O}(g^{-1}(k))}\right) n^{\mathcal{O}(1)}$.

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Notably when $g(r) = \Theta(r^2)$ and $h(w) = 2^{o(w^2)}$, this time is subexponential $\implies$ subexponential algorithms for vertex cover, feedback vertex set, dominating set,. . .

Locally Bounded Tree-Width

$$\forall v.\, \mathbf{tw}(G[N_r(v)]) \leq f(r)$$

Theorem

Every apex-minor-free graph of diameter D has treewidth in $\mathcal{O}(D)$.

Diameter-bounded **tw** $\approx$ locally-bounded **tw** (neighbourhood is a minor)

Separators

Theorem

Every H-minor-free graph G has **tw** in $\mathcal{O}\left(\sqrt{|V(G)|}\right)$.

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Proof: Number of vertices is r^2-bidimensional.

Consequence: Every vertex-weighted H-minor free graph has a separator of size $\mathcal{O}\left(\sqrt{|V(G)|}\right)$, which separates into two parts with weight at most $\frac{2}{3}$ of G
$\implies$ divide and conquer

Separation Property of Problems

1. can be solved for each connected component independently
2. there is a polynomial algorithm, which given a cut of the graph and optimal solutions for all connected components, computes solution for the whole graph which is not much greater
3. optimal solution of the whole graph is not much different from optimal solution of connected components from cut

(much $\approx$ size of the cut)

PTAS for BiD problems

Theorem

When a BiD problem with separation property can be approximated with constant factor in polynomial time it has a PTAS (all on H-minor-free graphs).

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When a BiD problem with separation property can be approximated with constant factor in polynomial time it has a PTAS (all on H-minor-free graphs).

- Recursively decompose graph into subgraphs, until approximate solutions are precise enough.
- When decomposing, approximate tree decomposition and choose one bag as a cut (which divides most evenly).
- Finally join solutions using separation property.

Open Problems

- find better bounds for grid minors in general graphs (lower bound $r^2 \log r$, conjecture r^3)
- generalize BiD (and resulting PTASs) for weighted problems

...

Summary

Every H-minor-free graph of $\mathbf{tw} = w$ has $G_{\Omega(w)\times\Omega(w)}$ as a minor. This can be used:

- prove that problem is FPT
- find subexponential FPTAs
- build PTASs

Bibliography

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