MA052 / A1

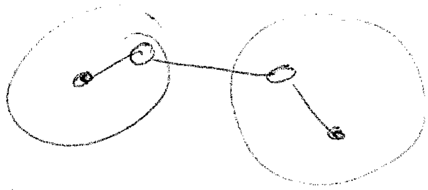
Half-integer terminy – we do not allow boundary to boundary edges

r-shallow $\sim$ distance $\leq 2r+1$ (podrozdělíme $2r$ vrcholy)

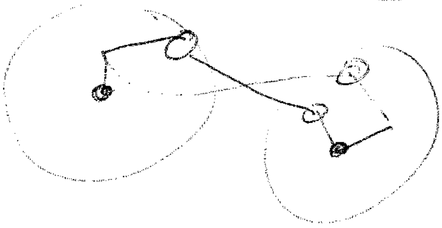
half-shallow

musí jít přímo doprostřed

1-shallow



$1\frac{1}{2}$ shallow



$\Rightarrow$ r-shallow for $r \in \frac{1}{2} \cdot \mathbb{N}$

Attributes of sparsity

1. edge density ($MAD(G) = \max_{H \subseteq G} \frac{2 \cdot \|H\|}{|H|}$)
2. Existence of cliques ($\omega(G)$ = clique size)
3. colouring ($\chi(G)$)

All of them are different.

$\omega \leq \chi \leq MAD$ (ω forces everything to be high)

MA052 / A2

Those measures coincide in the sparsity theory.

<u>FACT</u>: ~~???~~ $\frac{\|G\|}{|G|}$ - high $\Rightarrow \exists H \subseteq G$ such that $\delta(H)$ is high.

↳ "average degree" (not exactly, constant 2)

Proof: If $\deg(v) < (1-\varepsilon)\cdot\frac{\|G\|}{|G|}$, we remove vertex ~~?~~ v and repeat. At the end, we delete less than $|G|\cdot(1-\varepsilon)\frac{\|G\|}{|G|} = (1-\varepsilon)\|G\|$, left with more than $\varepsilon\|G\|$ edges, $\delta \geq (1-\varepsilon)\frac{\|G\|}{|G|}$.

□

Last time we said:

$$\frac{MAD}{2} \leq \text{Degeneracy} \leq MAD$$

<u>FACT</u>: $\omega(G) \leq \text{degeneracy}_G + 1$

$\chi(G) \leq$ degeneracy of $G + 1$

The point is to make theory which makes all of ~~????~~ those measures "equally strong".

MA052 / A3

Attribute GRAD (ad (1)) - it is about edge density.

DEF: minor resolution: For G a graph let $\underline{G \triangledown r}$ be the class of r-shallow minors of G.
($G \tilde{\triangledown} r$ for r-shallow topological minors).

percieve r as time:

$$G \to \underbrace{G \triangledown 0}_{\substack{\text{subgraphs} \\ \text{are allowed} \\ \text{MAD}}} \to G \triangledown \tfrac{1}{2} \to G \triangledown 1 \to \dots$$

av. degree ... Grad ...

more generally, $\mathcal{G} \triangledown r$ for gr. class $\mathcal{G}$.

DEF: GRAD $\nabla_r(\mathcal{G}) = \max \left\{ \frac{\|H\|}{|H|} \mid H \in \mathcal{G} \triangledown r \right\}$
(TOP-GRAD — grad for topological)

FACT $\frac{1}{2} \text{mad}(G) = \nabla_0(G) \leq \nabla_{1/2}(G) \leq \nabla_1(G) \leq \dots \nabla_\infty(G)$
$= \nabla G$

— this is related to the clique size

MAO52 / A4

DEF: $h(G) = \max\{\omega(H) : H \text{ minor of } G\}$

Hadwiger number = largest clique that can be formed as a minor

LEMMA: $\frac{h(G)-1}{2} \le \nabla(G) \le o\left(h(G)\sqrt{\log h(G)}\right)$

known: $d(H) > k \cdot \sqrt{\log k} \Rightarrow h(H)$ in k

The proof is in Diestel

$\Rightarrow$ this relates the clique size (as a minor) to Grad

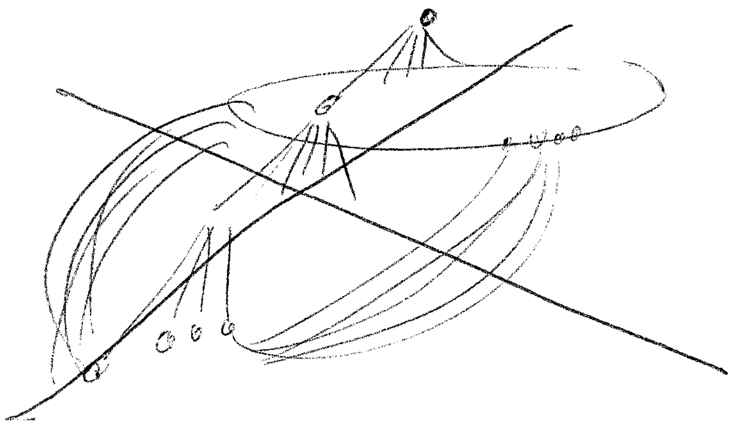
Thm: $\tilde{\nabla}_r(G) \le \nabla_r(G) \le 4 \cdot (4 \cdot \tilde{\nabla}_r(G))^{(r+1)^2}$

This means that Grad and TOP-Grad are ~~exponen~~ polynomially equivalent.

MA052 / A5

$\boxed{\text{GRAD and } \omega}$ ~~have relation without~~

~~$\nabla_r(G)$~~



Problem: $\nabla_r(G)$ is high, does it imply ~~$G \nabla r$ has large clique as a subgraph? not true~~ that there $\exists s$ such that $G \nabla (r+s)$ has a large clique. s is dependent of r, nebo možná ne

OR: $mad(G)$ high $\Rightarrow \exists s$ such that $G \nabla s$ has large clique

Sparsity: získat kliku je nejtěžší.

Otázka se vztahuje 2 věc?

$\boxed{\text{Grad and } \chi}$

~~Thm~~: $mad(G) > \Omega(c^2 \log c) \Rightarrow G \nabla \frac{1}{2}$ has graph
LEM H of $\chi(H) \geq c$.

MAC52 / A6

THM: $$\frac{\chi(G \,\tilde\nabla a) - 1}{2} \le \tilde\nabla_a(G) \le \sigma\left(\chi\left(G \,\tilde\nabla (2a+\tfrac{1}{2})\right)^{4}\right)$$

CONCLUSION

* measure density by ∇, χ, ω $\to$

$$\begin{array}{ccccc}
\nabla_0(G) & \le & \nabla_1(G) & \dots & \nabla(G) \\
\wr & & \wr & & \wr \\
\tilde\nabla_0(G) & \le & \tilde\nabla_1(G) & \dots & \tilde\nabla(G) \\
\sim & & \sim & & \sim \\
\chi(G\nabla 0) & & \chi(G\nabla 1) & \dots & \\
\omega(G\nabla 0) & \dots & \omega(G\nabla 1) & \dots & \omega(G\nabla\infty)
\end{array}$$

nic tu není $\cancel{\chi}$

Hustá třída ... v rezoluci dostaneme eventuelně vysokou barevnost, všechny hrany, všechny kliky ...

Řídká třída ... nikdy, ani s vysokým r, nedostaneme ~~větší~~ všechny kliky atd.