Bounded Expansion

Observation: From trichotomy, nowhere dense is somehow related to : $\forall t\ \nabla_t(C) = n^{o(1)}$

↳ we approach 1 from above.

Def: Class $\mathcal{C}$ has bounded expansion (by function $\mathbb{N}/2 \to \mathbb{R}$) if $\forall t\ \nabla_t(\mathcal{C}) \leq f(t)$.

"At every level, the average degree remains constant".

→ We have "nowhere dense" of $\mathcal{C}$.

↳ supremum of clique size is bounded

We can find an example of nowhere dense without bounded expansion.

Prop. $\mathcal{C}$ has bounded expansion $\Leftrightarrow \forall t\ \sup\limits_{G \in \mathcal{C}\nabla_t} \chi(G) < \infty$

Idea: "$\Rightarrow$": For every graph, G is of bounded grad, then degenerate and χ is bounded.

"$\Leftarrow$": "high degrees imply t-subdivision of high χ".

i.e., if $\mathcal{C}$ is not bounded exp, in depth $2t$, we have unbounded χ

MA052 C2

Example:

unbounded expansion and nowhere dense

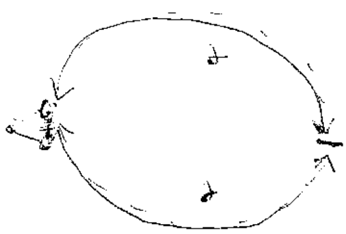
$\mathcal{E} = \{ G : \delta(G) \leq \underline{\text{girth}}(G) \}$. Such graphs do exist

↳ length of shortest cycle

↳ in grad, the degrees grow ⇒ unbounded expansion is clear

↳ nowhere dense: We need to get a clique to get somewhere dense. It is not possible: if $K_3 \in G \nabla t$ for $G \in \mathcal{E}$, then girth $\leq 2t$ which by definition $\Delta G \leq$ girth $\leq 2t$ and so $\omega(G \nabla t) < \underline{\Delta(G)^{2t}}$

↳ value does not matter, but it is bounded.



we want to contract the long shortest cycle to a triangle

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"Locally" Bounding Something (*)

Instead of bounding something, we just bound it locally. We can start from bounded TW.

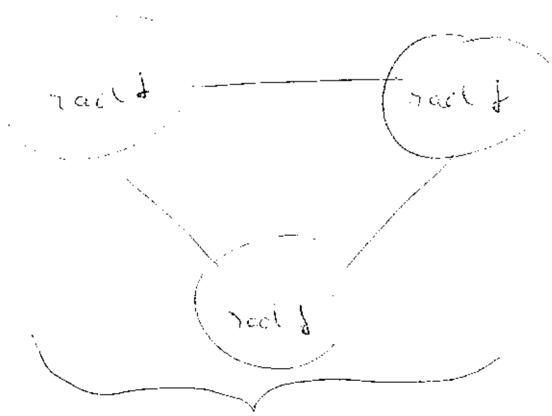
Planar graphs do not have bounded TW, but have locally bounded TW.

Informally, the property * is bounded in every radius r induced subgraph with bound that may depend on r.

Example: Locally nowhere dense $\iff$ nowhere dense.

Proof: somewhere dense $\Rightarrow$ locally somewhere dense.

At some resolution, we get big clique of size $n \to \infty$



diameter $\leq 2t + 2t + 1$

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Def: Locally bounded expansion $\mathcal{C}$ and corollary if $\forall r \in \mathbb{N}\ \forall G \in \mathcal{C}$ of radius r then $\nabla_r(G) \leq f(r, r)$ for all r.

The class $\mathcal{E}$ before is of locally bounded expansion.

Proposition: Locally bounded expansion implies nowhere dense.

Class $\mathcal{E}$ with a vertex connected to everything ("apex vertex") produces locally unbounded expansion and nowhere dense.

Tree-depth - more on it

Prop: ~~$\mathrm{td}(G)$~~ $\mathrm{tw}(G) \leq \mathrm{pw}(G) \leq \mathrm{td}(G) - 1 \leq (\mathrm{tw}$ $\cdot \log$

visible — invisible — visible, no lifting

Proof $\mathrm{td}(T) \leq \log n$ by induction. Find $x \in V(T)$ such that all components in $T - x$ are size at most $\frac{n}{2}$. Land a cop at x and recourse. Generally, "multiply" this game by the bag size in G.

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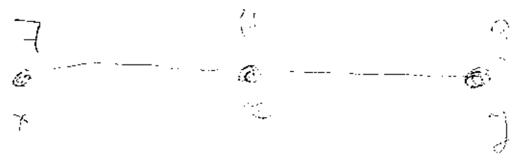
Tree depth - no long paths

~~Def~~

Def: (Vertex ranking of G) Vertex colouring c by colours $\{1..d\}$ such that $\forall$ path $P \subseteq G$ with ends x, y there exists z such that $c(z) > c(x), c(y)$ when $c(x) = c(y)$



Def. (Centered colouring of G): Any connected subgraph F of G, there exists a colour occuring precisely once in F. (That colour is the center).

Prop
Both of them are normal colourings.

Prop: Both vertex ranking and center colouring are equal to tree-depth (colouring numbers and height of tree-depth decomposition; equal not asymptotically, but precisely).

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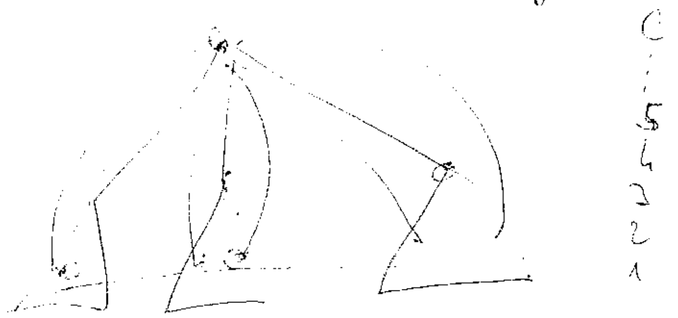
Proof outline

Trees' depth decomposition ⇒ all edges vertical (by



Vertex ranking

This is also centred colouring - every subgraph has at most one top vertex.