MAGV2 C1

p - tree depth colouring

Last time, we said that p-centered colouring is equal to tree depth (or tree-width?)

Motivation: colouring χ $\xrightarrow{\text{generalization}}$ acyclic colouring χ_a

Union of every 2 classes is acyclic graph

Union of any 3 classes would be a $tw = 2$

From here, we can go to p-tree-width colouring $\chi_{p\text{-}tw}$.

Def: χ_{ptw} colouring of G = minimal c such that there is partition $(X_1 \dots X_c)$ of $V(G)$ such that $\exists I \subseteq \{1 \dots c\}$, $|I| \leq p$, $G[\bigcup_I X_i]$ has $tw \leq |I|-1$.

This reads as any J ($I = J$) classes induce colouring of tree-with at most $|J|-1$.

Why is it useful? For Local properties!

MA052 C8

Another motivation (Nešetřil):

colouring χ $\longrightarrow$ star colouring χ^*, χ_s

Union of any two classes is a forest of stars

$\Rightarrow$ more restricted

↳ also defineable as no path in form

Red - Blue - Red - Blue (length 4).

Stars have tree-depth = 2, i.e. we can generalize to p-tree-depth colouring χ^{p} (variable)

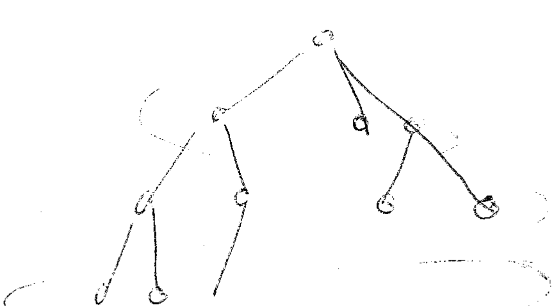
Def: $\chi_p(G) = \min c$ such that ...

$G[\cup_J X_i]$ has td $\leq |J|$.

Proposition $\chi(G) = \chi_1(G) \leq \chi_2(G) \leq \dots \leq \chi_{td(G)}(G)$

every class is independent

this is the tree-depth



levels are the colours to take

MA052 C3

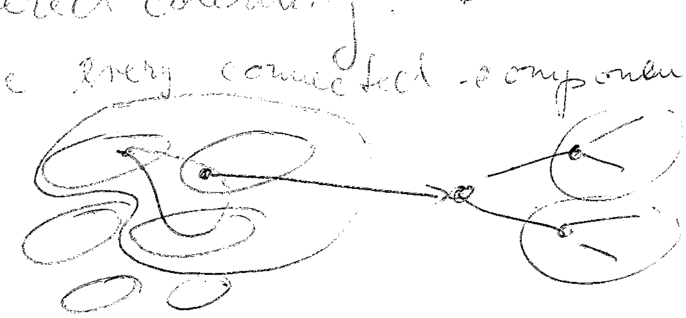
<u>Question</u>: Is the sequence of χ_p bounded on the whole class of graphs? (instead of a single graph).

<u>Def</u>: Centered colouring - in every connected subgraph, ~~every~~ at least one colour is used only once.

<u>Def</u>: p-centered colouring of G is a proper colouring such that in any connected subgraph $H \subseteq G$, either H has $\geq p$ colours or some colour occurs exactly once in H.

<u>Prop</u>: $\chi_p(G) \leq$ min # colours in a $(p+1)$-centered colouring. (p is number of classes, not colours)

<u>Proof</u>: take $(p+1)$-centered colouring. Take at most p classes. Take every connected component, this has p colours and



thus it has some colour only once. Recursively branch, td $\leq j$.

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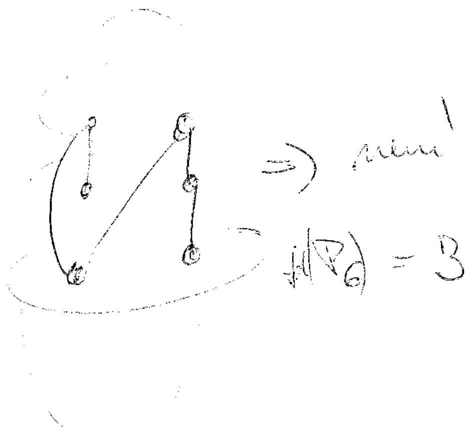
On the other hand, take p-tree-depth colouring.

~~If it lists more than p classes~~

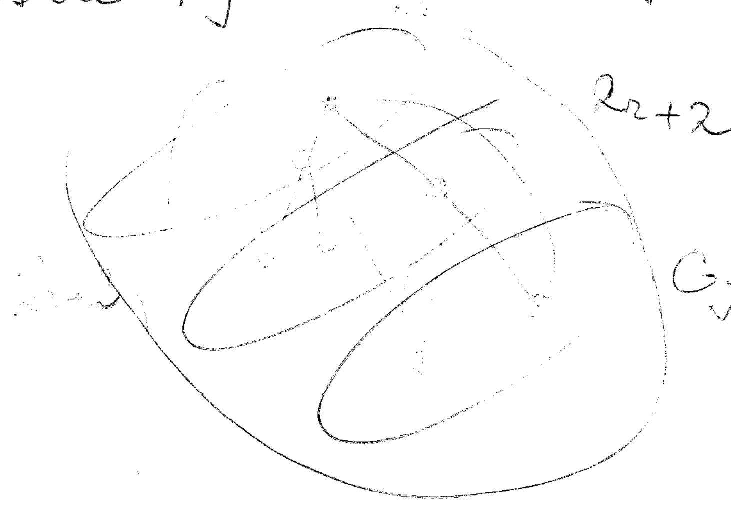
Neplati!



We want to relate these numbers to bounded expansion and grad:

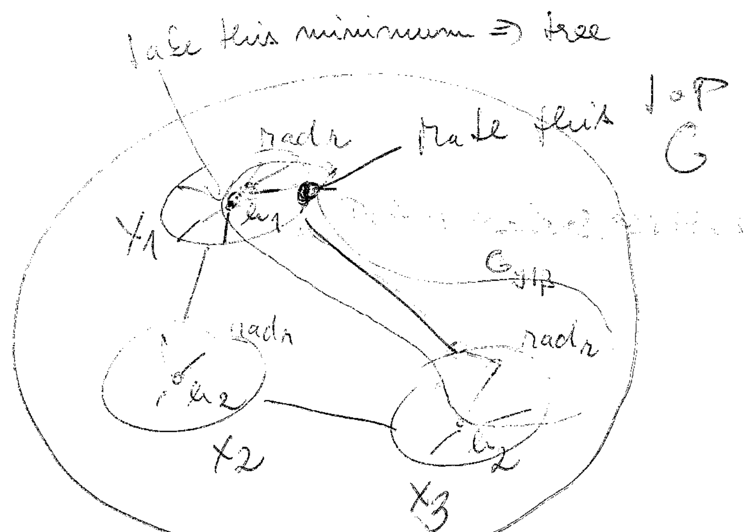
Prop: $\nabla_r(G) \le (2r+1)\binom{\chi_{r+2}(G)}{2r+2}$ If this is bounded in every r, then the grad is bounded.

Proof: Use $\chi_{2r+2}(G)$ colours on G. Denote by G_J, for $J \subseteq \{1 \dots \chi_{2r+2}\}$ and $|J| = 2r+2$, G_J is the union of these classes, and G_J is a subset of closure Y_J - model of td decomposition



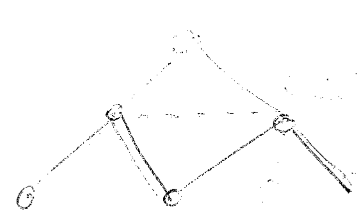
MA052 C5

Consider ~~...~~ $H \in G \triangledown r$ which may be bad (very dense). We want to show that H does not have too many edges. Focus on how H resides in G.



$V(H) = \{b_1, b_2, \ldots\}$

The red path is contained in corresponding G_y, i.e. $G_{y_{1,3}}$. This $G_{y_{1,3}}$ is in a tree $T_{1,3}$, it is not a subpath, but surely one vertex is the top one as this cannot happen ✱



Make the top vertex Top

NAOS2 C6

Top is an ancestor of h_1 $\cancel{\|H\|} \leq |H| \cdot \nabla_r(G) \leq |H| \cdot \frac{\|H\|}{|H|} =$

upper bound for edge density

$$\leq \cancel{\quad} \sum_{I,\ |I| = 2r+2} \sum_{i=1}^{|H|} \#\text{ancestors of } h_i \text{ belonging to } \tilde{Y}_I \cap X_y$$

$$\leq \sum_{I,\ |I| = 2r+2} \sum_{i=1}^{|H|} \#\text{ of ancestors of } x_i \text{ in } \tilde{Y}_I$$

$$\leq \binom{\chi_{2r+2}(G)}{2r+2} \cdot |\cancel{H}| \cdot (2r+1)$$

throw away, we have the grad

$\square$

Now, you would like to have the opposite – low grad implies low col-colouring.

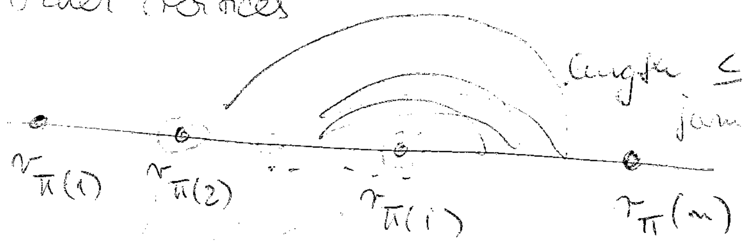
MA 052 C7

Def.: Weak colouring of G on $V(G) = \{v_1 \dots \cancel{v_n}\}$

$$\mathrm{wcol}_k(G) = 1 + \min_{\pi \in S_n} \max_i |Q_k(G, \pi, i)|$$

permutation ($\pi \in S_n$), positions (i)

Q_k: order vertices



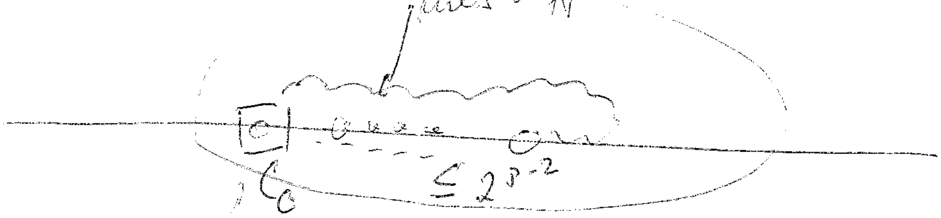
length $\leq k$, path that jumps to the left most vertex, that belongs to Q_k

– set of vertices on the left of $v_{\pi(i)}$ ~~connect~~ are called "weakly accessible from $v_{\pi(i)}$"

THM: $\mathrm{wcol}_{2^{p-2}}(G) \leq m$, then there is p-centered colouring of m colours.

Proof

uses different colour



colour used ~~at~~ once

By taking half of the path, do binary search, all vertices have weak access, thus different colours.