Digital Signal Processing

Discrete Sequences and Systems

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Discrete Sequences and Their Notation

- Signal processing
  - Science of analyzing time-varying physical processes
  - Continuous signal
    - Continuous in time
    - Continuous range of amplitude values
    - Analog (continuous) signal processing
  - Discrete-time signal
    - Time variable is quantized
    - Signal amplitude is quantized
      - Because we represent all digital quantities with binary numbers, there's a limit to the resolution
    - Digital signal processing

Discrete Sequences and Their Notation

- Example
  - A continuous sinewave
  - Peak amplitude of 1
  - Frequency f_o

$$x(t) = \sin(2\pi f_o t)$$

  - f_o is measured in hertz (Hz) = cycles/second
  - t representing time in seconds
  - $f_o t$ has dimensions of cycles
  - $2\pi f_o t$ is an angle measured in radians

Discrete Sequences and Their Notation

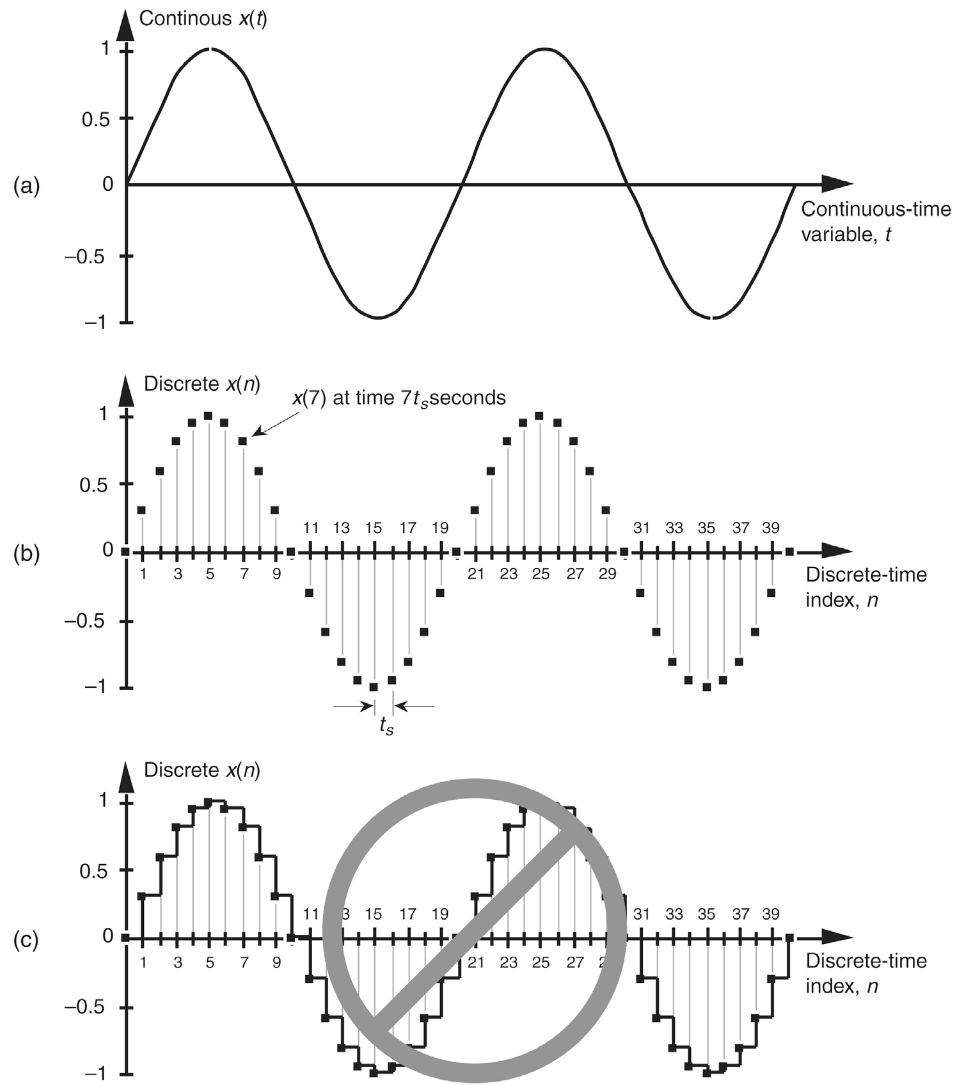

Figure 1-1 A time-domain sinewave: (a) continuous waveform representation; (b) discrete sample representation; (c) discrete samples with connecting lines.

Discrete Sequences and Their Notation

- Fig. 1-1
  - Continuous sinewave → *sample* it once every t_s seconds using an analog-to-digital converter
  - Variable *t* in (a) is continuous
  - *Index* variable *n* in (b) is discrete and can have only integer values
  - *x*(*n*) is a discrete-time sequence of individual values
    - There *is* nothing between dots of *x*(*n*)

$$x(t) = \sin(2\pi f_o t) \rightarrow x(n) = \sin(2\pi f_o n t_s)$$

  - *x*(*t*) and *x*(*n*) are referred to as *time-domain* signals

Discrete Sequences and Their Notation

- Discrete system
  - A collection of hardware components, or software routines, that operate on a discrete-time signal sequence

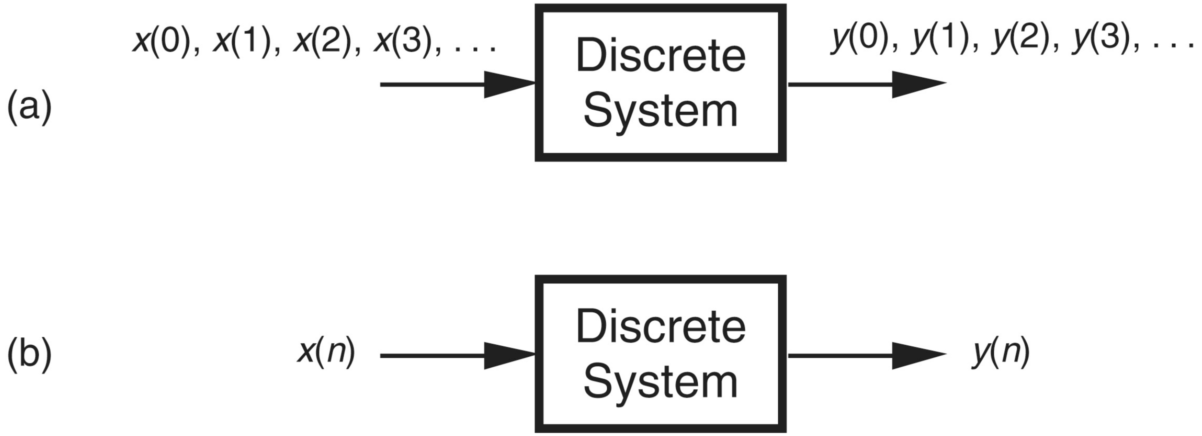

Figure 1–2 With an input applied, a discrete system provides an output: (a) the input and output are sequences of individual values; (b) input and output using the abbreviated notation of $x(n)$ and $y(n)$.

  - E.g., $y(n) = 2x(n) - 1$

Discrete Sequences and Their Notation

- Given samples of a discrete-time sinewave (e.g., Fig. 1-1(b)), find frequency of waveform they represent
  - Possible to say sinewave repeats every 20 samples
  - Not possible to find exact sinewave frequency
    - We need sample period t_s to determine absolute frequency of discrete sinewave
  - If t_s = 0.05 milliseconds/sample

$$\text{sinewave period} = \frac{20 \text{ samples}}{\text{period}} \times \frac{0.05 \text{ milliseconds}}{\text{sample}} = 1 \text{ milliseconds}$$

  - Sinewave's frequency = 1/(1 ms) = 1 kHz

Discrete Sequences and Their Notation

- Frequency domain
  - To represent frequency content of discrete time-domain signals
  - Called *spectrum*

Discrete Sequences and Their Notation

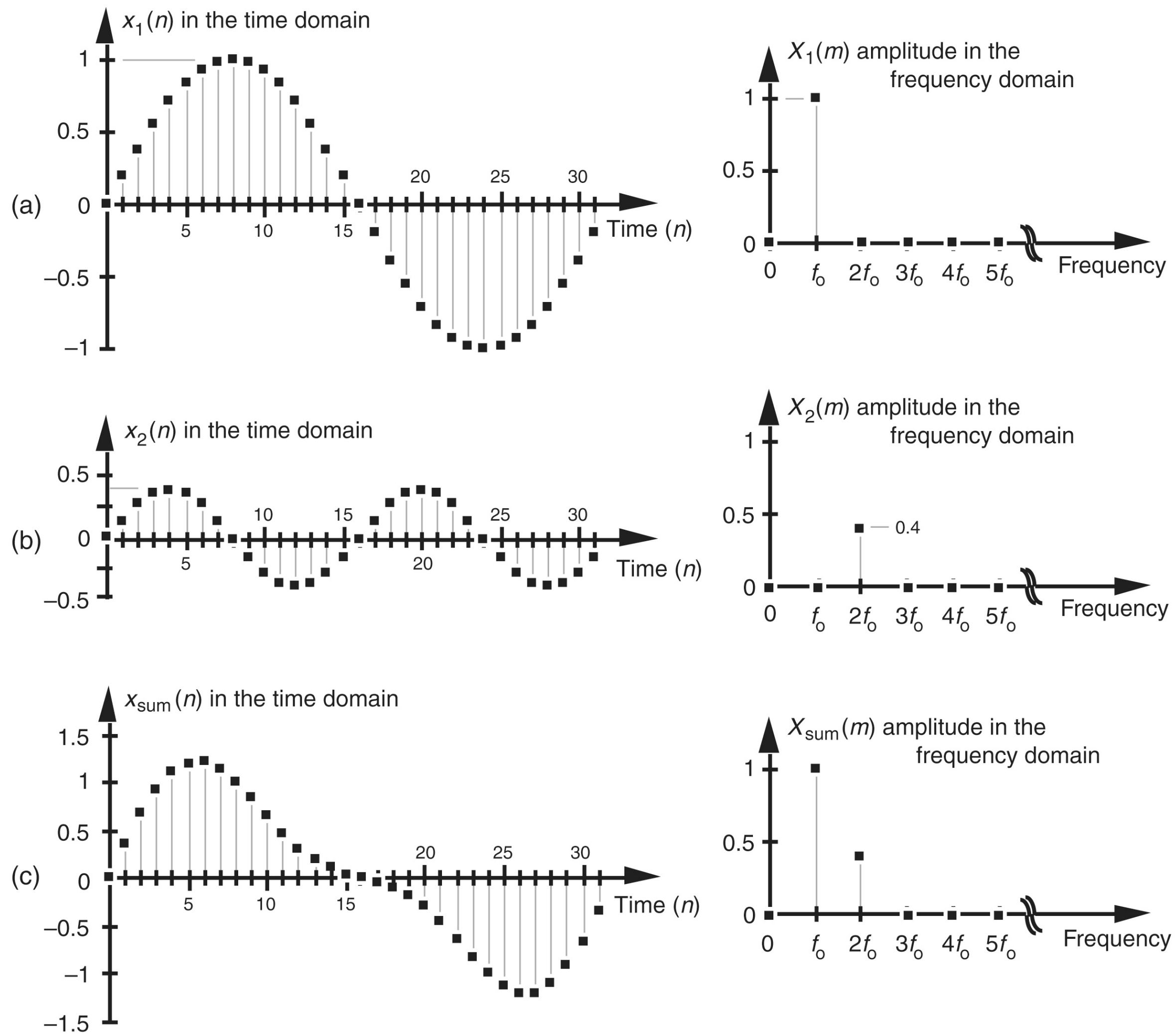

Figure 1–3 Time- and frequency-domain graphical representations: (a) sinewave of frequency f_o; (b) reduced amplitude sinewave of frequency $2f_o$; (c) sum of the two sinewaves.

Discrete Sequences and Their Notation

- Fig. 1-3

$$x_{sum}(n) = x_1(n) + x_2(n) = \sin(2\pi f_o n t_s) + 0.4 \times \sin(2\pi 2 f_o n t_s)$$

  - $x_{sum}(n)$ has a frequency component of f_o Hz and a reduced-amplitude frequency component of $2f_o$ Hz
  - Because $x_1(n)$ + $x_2(n)$ sinewaves have a phase shift of zero degrees relative to each other, no need to depict this phase relationship in $X_{sum}(m)$
    - In general, phase relationships in frequency-domain sequences are important

Signal Amplitude, Magnitude, Power

- Amplitude of a variable
  - Measure of how far, and in what direction, that variable differs from zero
  - Can be either positive or negative
- Magnitude of a variable
  - Measure of how far, regardless of direction, its quantity differs from zero
  - Always positive

Signal Amplitude, Magnitude, Power

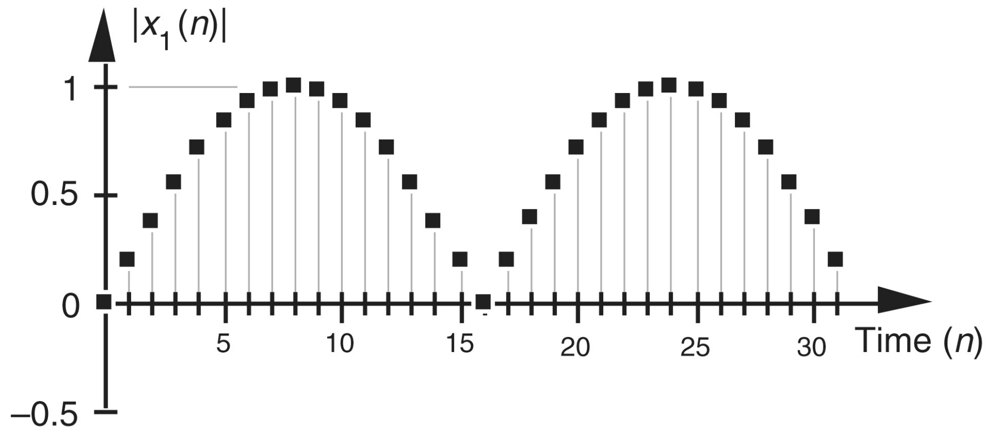

Figure 1–4 Magnitude samples, $|x_1(n)|$, of the time waveform in Figure 1–3(a).

Signal Amplitude, Magnitude, Power

- In frequency domain, we are often interested in power level of signals
  - Power of a signal is proportional to its amplitude (or magnitude) squared
  - Assuming proportionality constant is one, power of a sequence in time or frequency domains are

$$x_{pwr}(n) = |x(n)|^2, \qquad X_{pwr}(m) = |X(m)|^2$$

  - Often we want to know the difference in power levels of two signals in frequency domain
    - Because of squared nature of power, two signals with moderately different amplitudes will have a much larger difference in their relative powers

Signal Amplitude, Magnitude, Power

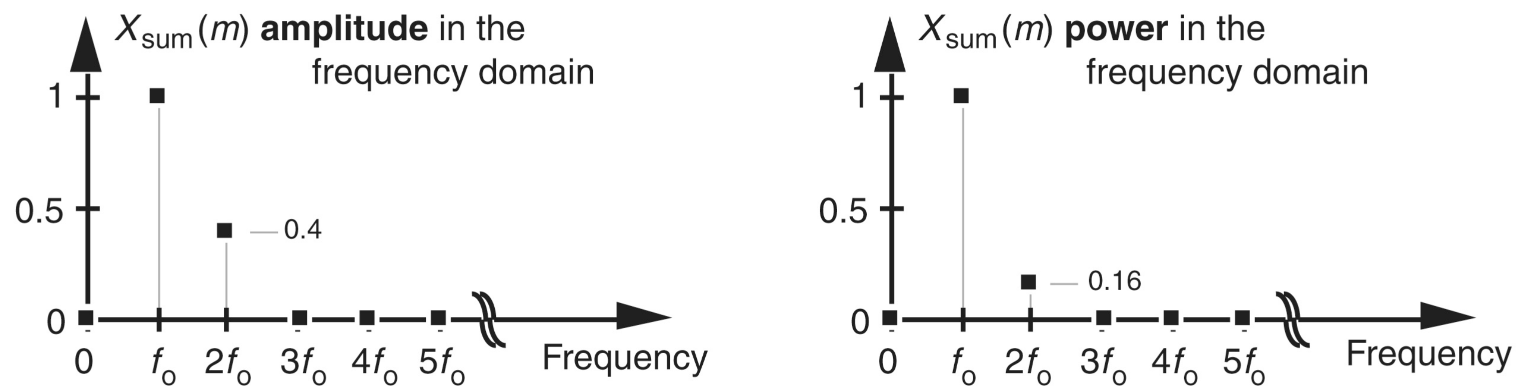

Figure 1–5 Frequency-domain amplitude and frequency-domain power of the $x_{sum}(n)$ time waveform in Figure 1–3(c).

- Because of their squared nature, plots of power values often involve showing both very large and very small values on same graph
  - To make these plots easier to generate and evaluate, decibel scale is usually employed

Signal Processing Operational Symbols

- Block diagrams
  - Are used to graphically depict the way digital signal processing operations are implemented
  - Comprise an assortment of fundamental processing symbols

Signal Processing Operational Symbols

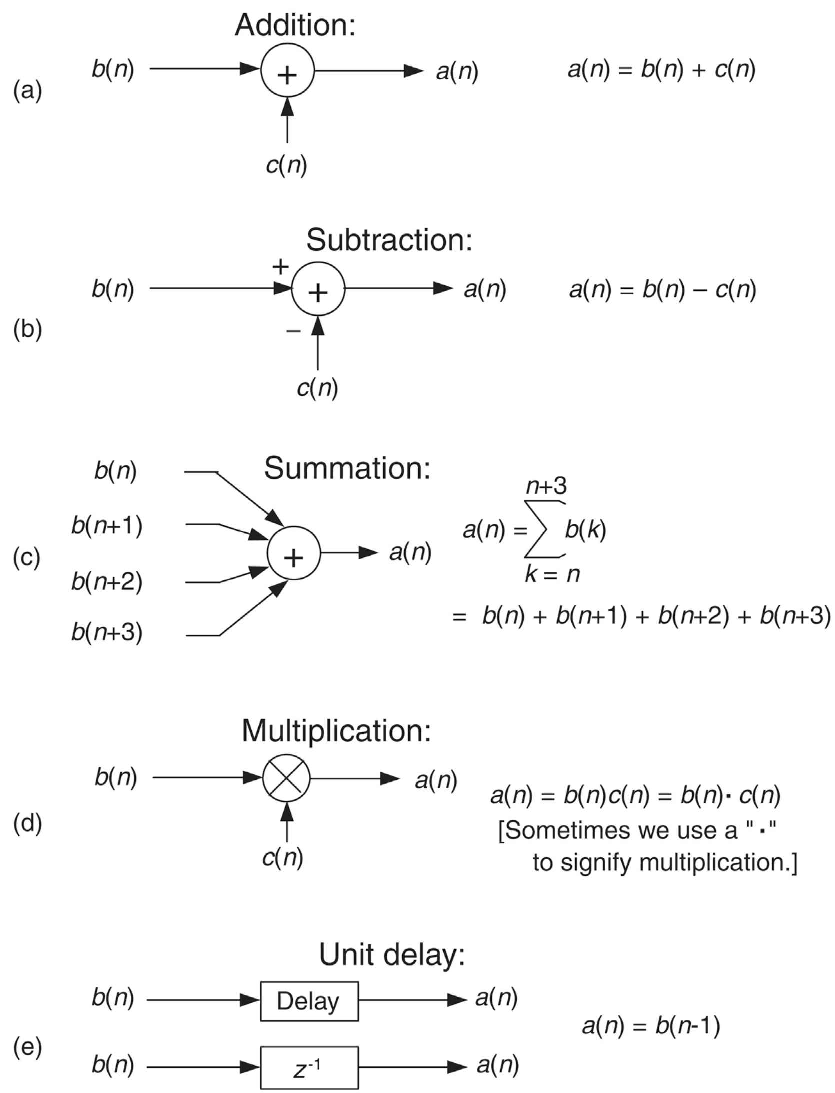

Figure 1–6 Terminology and symbols used in digital signal processing block diagrams.

Discrete Linear Time-Invariant Systems

- Linear time-invariant (LTI) systems
  - Vast majority of discrete systems used in practice are LTI systems
  - LTI systems are very accommodating when it comes to their analysis
    - We can use straightforward methods to predict performance of any digital signal processing scheme as long as it’s linear and time invariant

Discrete Linear Systems

- Linear
  - A linear system's output resulting from a sum of individual inputs is superposition (sum) of individual outputs

$$x_1(n) \xrightarrow{\text{results in}} y_1(n)$$

$$x_2(n) \xrightarrow{\text{results in}} y_2(n)$$

$$x_1(n) + x_2(n) \xrightarrow{\text{results in}} y_1(n) + y_2(n)$$

  - Also, if inputs are scaled by constant factors c_1 and c_2, output sequence parts are scaled by those factors too

$$c_1 x_1(n) + c_2 x_2(n) \xrightarrow{\text{results in}} c_2 y_1(n) + c_2 y_2(n)$$

Discrete Linear Systems

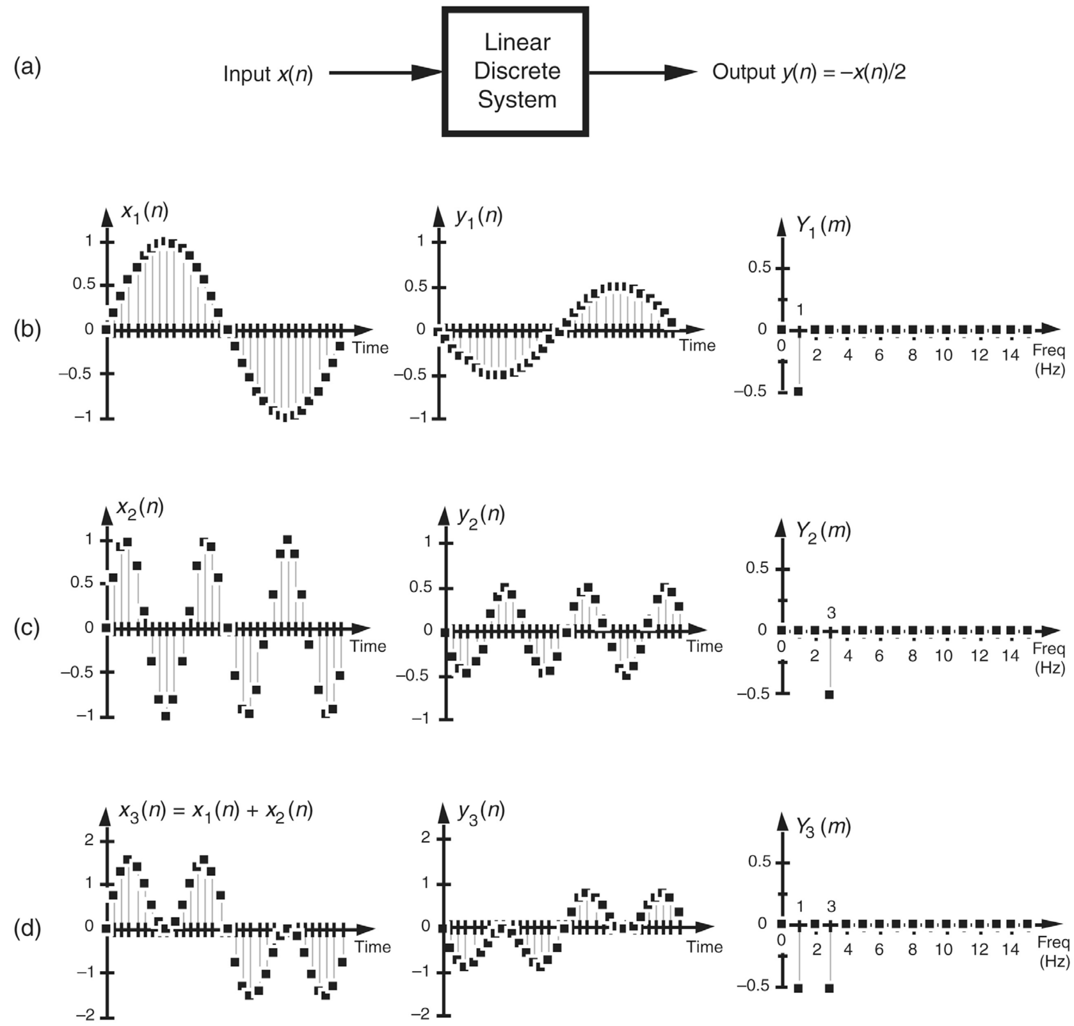

Figure 1–7 Linear system input-to-output relationships: (a) system block diagram where $y(n) = -x(n)/2$; (b) system input and output with a 1 Hz sinewave applied; (c) with a 3 Hz sinewave applied; (d) with the sum of 1 Hz and 3 Hz sinewaves applied.

Discrete Linear Systems

- Linearity in Fig. 1-7(d)
  - $x_3(n)$ input sequence is sum of a 1 Hz sinewave and a 3 Hz sinewave
  - Thus $y_3(n)$ is sample-for-sample sum of $y_1(n)$ and $y_2(n)$
  - Also output spectrum $Y_3(m)$ is sum of $Y_1(m)$ and $Y_2(m)$

Discrete Linear Systems

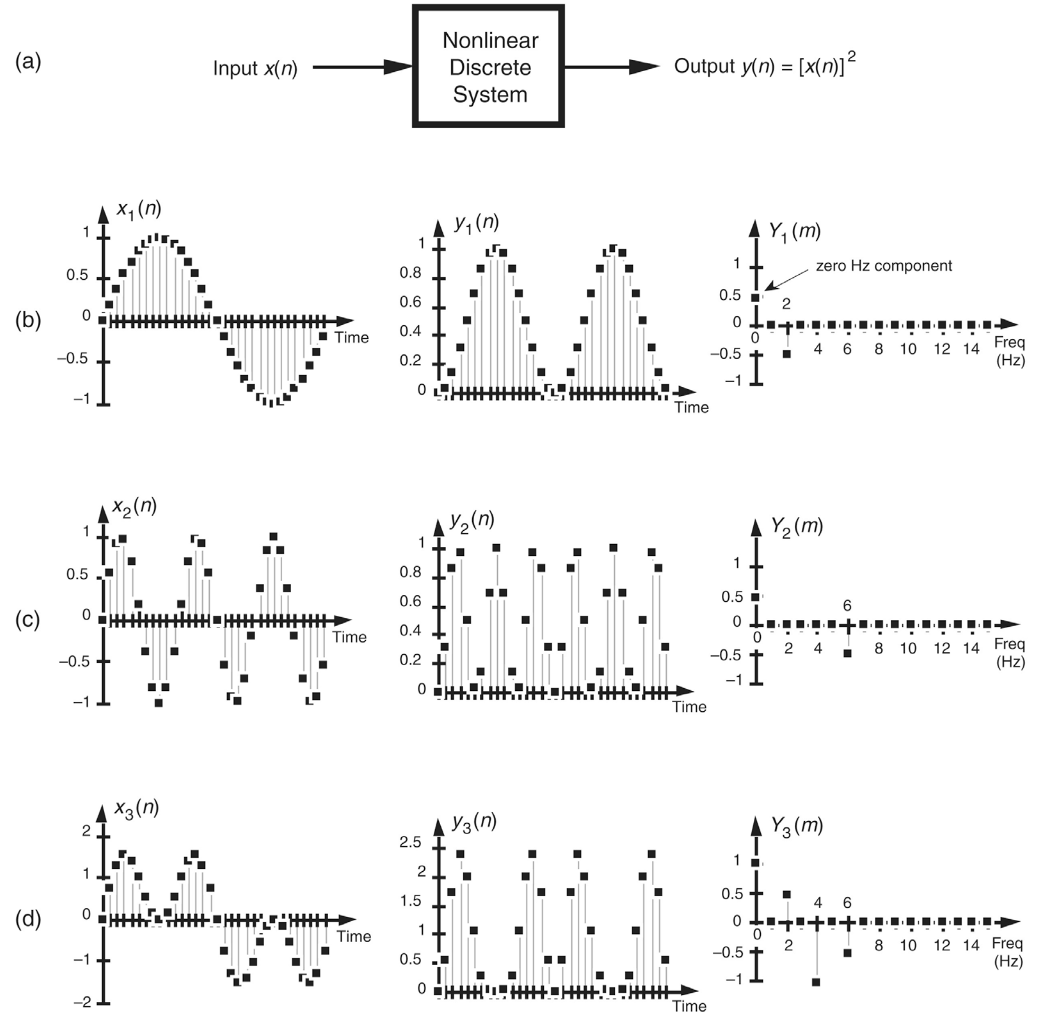

Figure 1–8 Nonlinear system input-to-output relationships: (a) system block diagram where $y(n) = (x(n))^2$; (b) system input and output with a 1 Hz sinewave applied; (c) with a 3 Hz sinewave applied; (d) with the sum of 1 Hz and 3 Hz sinewaves applied.

Discrete Linear Systems

- Fig. 1-8(b)

$$x_1(n) = \sin(2\pi f_o n t_s) = \sin(2\pi \times 1 \times n t_s)$$

$$y_1(n) = [x_1(n)]^2 = \sin(2\pi \times 1 \times n t_s) \times \sin(2\pi \times 1 \times n t_s)$$

$$\sin(\alpha) \times \sin(\beta) = \frac{\cos(\alpha - \beta)}{2} - \frac{\cos(\alpha + \beta)}{2}$$

$$y_1(n) = \frac{\cos(2\pi \times 1 \times n t_s - 2\pi \times 1 \times n t_s)}{2} - \frac{\cos(2\pi \times 1 \times n t_s + 2\pi \times 1 \times n t_s)}{2}$$

$$= \frac{\cos(0)}{2} - \frac{\cos(4\pi \times 1 \times n t_s)}{2} = \frac{1}{2} - \frac{\cos(2\pi \times 2 \times n t_s)}{2}$$

  - $y_1(n)$ is a cosine wave of 2 Hz and a peak amplitude of −0.5, added to a constant value (zero Hz) of 1/2
- Fig. 1-8(c)
  - $y_2(n)$ contains a zero Hz and a 6 Hz component

Discrete Linear Systems

- Fig. 1-8(d)
  - $x_3(n)$ comprises sum of a 1 Hz and a 3 Hz sinewave

$$a = 1\ Hz \text{ sinewave, } b = 3\ Hz \text{ sinewave} \rightarrow (a+b)^2 = a^2 + 2ab + b^2$$

$$a^2 \rightarrow zero\ Hz \text{ and } 2\ Hz$$

$$b^2 \rightarrow zero\ Hz \text{ and } 6\ Hz$$

$$\begin{aligned} 2ab &= 2\sin(2\pi \times 1 \times nt_s) \times \sin(2\pi \times 3 \times nt_s) \\ &= \frac{2\cos(2\pi \times 1 \times nt_s - 2\pi \times 3 \times nt_s)}{2} - \frac{2\cos(2\pi \times 1 \times nt_s + 2\pi \times 3 \times nt_s)}{2} \\ &= \cos(2\pi \times 2 \times nt_s) - \cos(2\pi \times 4 \times nt_s) \end{aligned}$$

$$2ab \rightarrow 2\ Hz \text{ and } 4\ Hz$$

  - Two additional sinusoids are present in $y_3(n)$ because of system’s nonlinearity, a 2 Hz cosine wave (amp=+1), a 4 Hz cosine wave (amp=−1)

Time-Invariant Systems

- Time-invariant system
  - A time delay (or shift) in input sequence causes an equivalent time delay in system's output sequence

$$x(n) \xrightarrow{results\ in} y(n)$$

$$x^{'}(n) = x(n+k) \xrightarrow{results\ in} y^{'}(n) = y(n+k)$$

    - *k* is some integer representing *k* sample period time delays
  - For a system to be time invariant, above equation must hold true for any integer value of *k* and any input sequence

Time-Invariant Systems

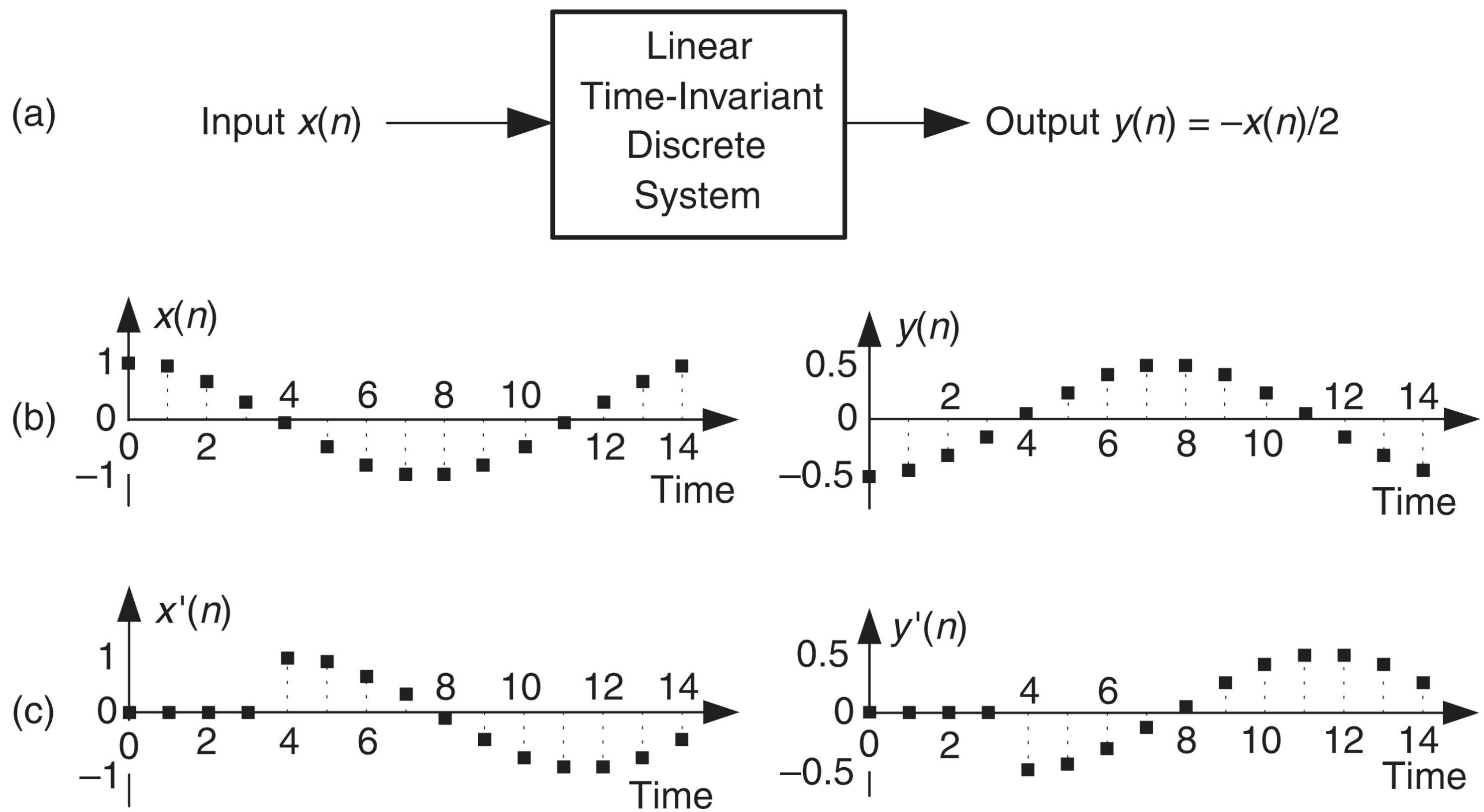

Figure 1–9 Time-invariant system input/output relationships: (a) system block diagram, $y(n) = -x(n)/2$; (b) system input/output with a sinewave input; (c) input/output when a sinewave, delayed by four samples, is the input.

Time-Invariant Systems

- Fig. 1-9
  - Input sequence $x'(n)$ is equal to sequence $x(n)$ shifted to right by $k = -4$ samples

$$x'(n) = x(n-4)$$

  - System is time invariant because $y'(n)$ output sequence is equal to $y(n)$ sequence shifted to right by four samples

$$y'(n) = y(n-4)$$

Commutative Property of LTI Systems

- LTI systems have a useful commutative property
  - Their sequential order can be rearranged with no change in their final output

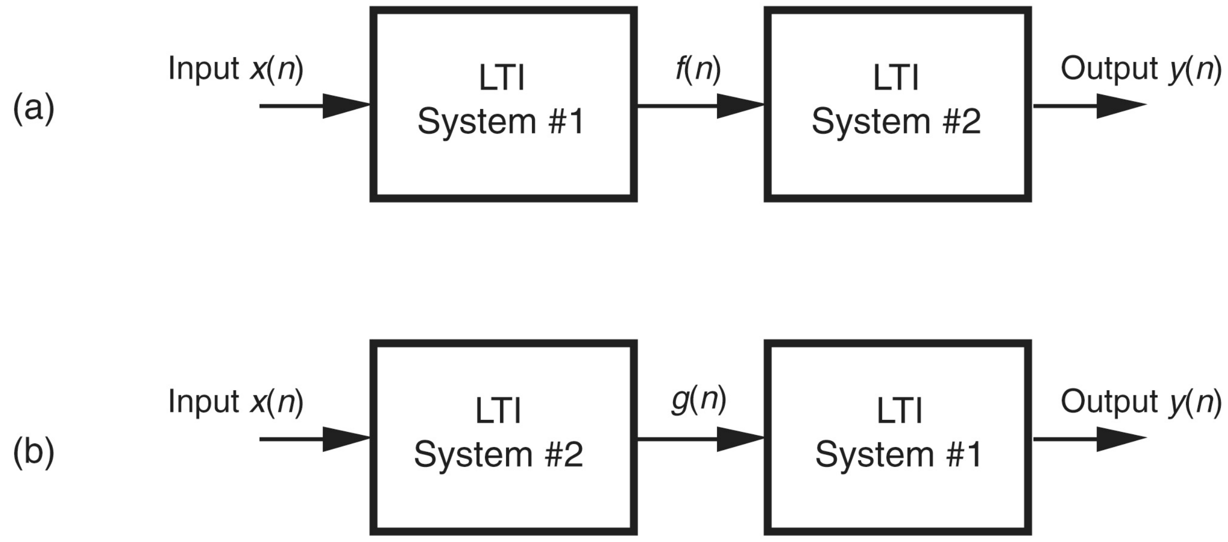

Figure 1–10 Linear time-invariant (LTI) systems in series: (a) block diagram of two LTI systems; (b) swapping the order of the two systems does not change the resultant output $y(n)$.

Analyzing LTI Systems

- Unit impulse response of an LTI system
  - System's time-domain output sequence when input is a single unity-valued sample (unit impulse) preceded and followed by zero-valued samples
- System's unit impulse response completely characterizes the system

Analyzing LTI Systems

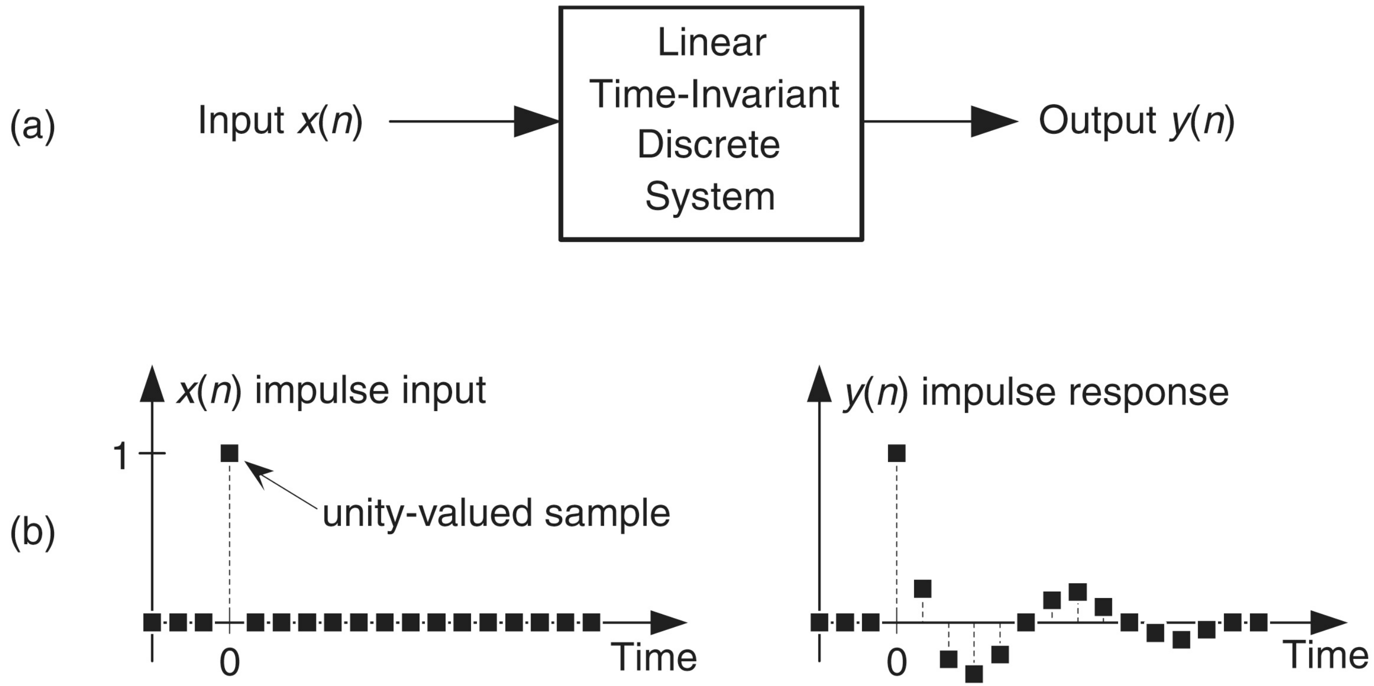

Figure 1-11 LTI system unit impulse response sequences: (a) system block diagram; (b) impulse input sequence $x(n)$ and impulse response output sequence $y(n)$.

Analyzing LTI Systems

- Knowing impulse response, we can determine system’s output for any input
  - Output is equal to *convolution* of input sequence and system’s impulse response
  - Moreover, we can find system’s *frequency response* by taking *discrete Fourier transform* of that impulse response

Analyzing LTI Systems

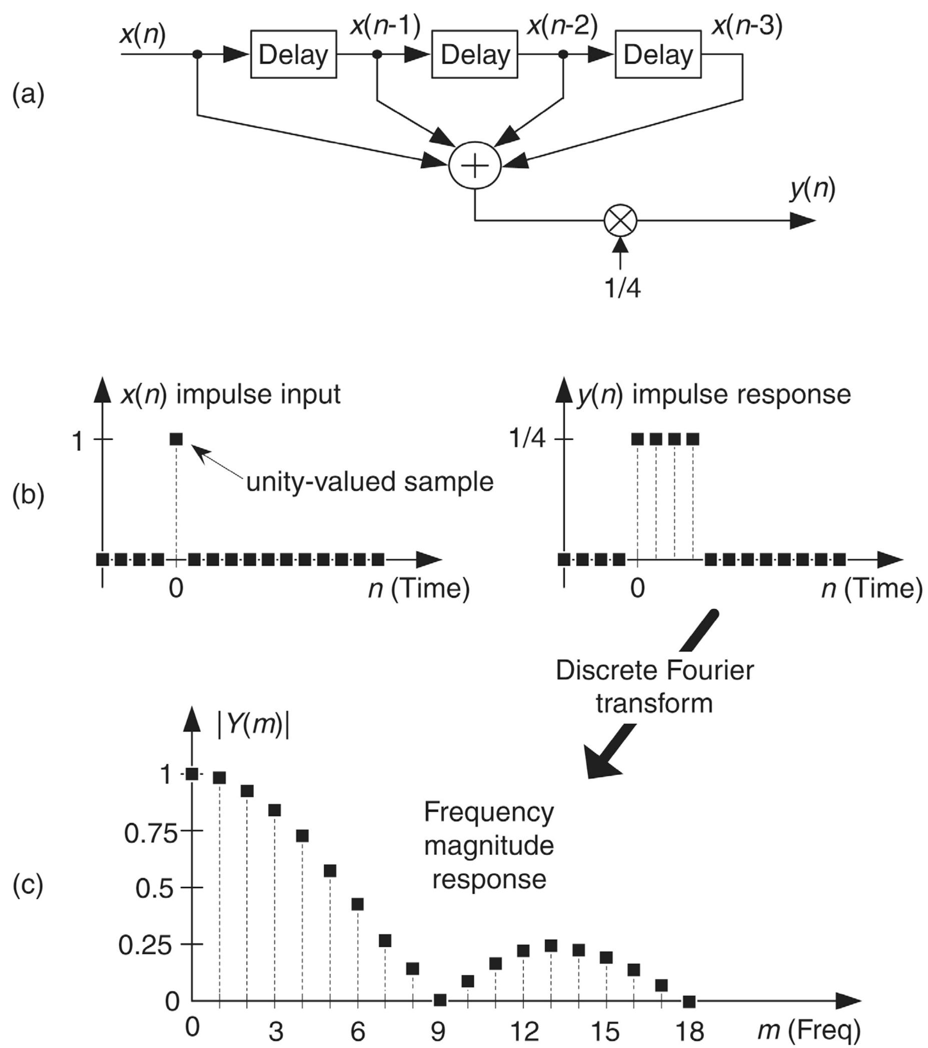

Figure 1-12 Analyzing a moving averager: (a) averager block diagram; (b) impulse input and impulse response; (c) averager frequency magnitude response.

Analyzing LTI Systems

- Fig. 1-12
  - A 4-point moving averager

$$y(n) = \frac{1}{4}[x(n) + x(n-1) + x(n-2) + x(n-3)] = \frac{1}{4}\sum_{k=n-3}^{n} x(k)$$

  - Frequency magnitude response plot shows that moving averager has characteristic of a lowpass filter
    - Averager attenuates (reduces amplitude of) high-frequency signal content applied to its input