JORDAN AND KURATOWSKI THEOREMS

Claim 1 Every planar graph has a polygonal plane embedding.

Claim 2 Every simple closed polygonal line divides the plane into 2 "pieces".

Idea 2 To distinguish inner and outer point, shoot a ray from it and count the number of intersections with line segments.

Those two areas are connected.

Claim 3 Every simple non-closed polygonal line does not divide ~~it~~ the plane.

Stronger 3 Every polygonal embedding of a tree has one face.

Proof 3 Let T be a tree with straight edges (otherwise add vertices).

~~For sake of a contradiction~~, let T be the smallest tree that divides a plane.

Erase vertex of degree one (every tree has one), so the resulting tree ~~in~~ doesn't divide the plane (it is smaller).

So every two points can be connected by a polygonal line, which crosses only the deleted edge. We can locally redraw the line to avoid this edge - CONTRADICTION!

Claim 4 For any polygonal embedding of a connected graph G in the plane we have

$$|V(G)| + \#faces - |E(G)| = 2.$$

Proof 4 By induction on $|E(G)| - |V(G)| + 1$.

Case 0 : G is a tree $\Rightarrow n + 1 - (n-1) = 2$

Case $s > 0$: So G contains some cycle, so by claim two there is an edge e and two points on different faces (different sides of e). If we erase e, this two faces will be merged.

Hence $\#faces(G-e) \leq \#faces(G) - 1$.

By induction ~~then~~ $n + \#faces(G-e) - |E(G-e)| = 2$,

$$n + \#faces(G-e) - |E(G)| - 1 = 2.$$

We can show $\#faces(G-e) \geq \#faces(G) - 1$ (homework)

and the claim follows.

Claim 5: $K_{3,3}$ is not planar.

Proof 5: For sake of contradiction assume $K_{3,3}$ is planar.

So it has a polygonal plane embedding.

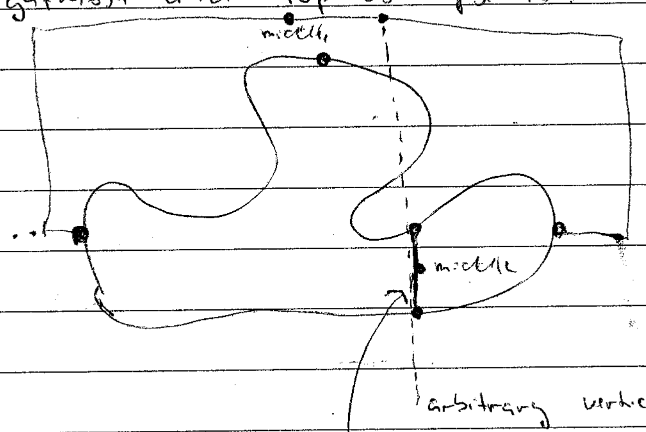
So $\#\text{faces}(K_{3,3}) = 9 + 2 - 6 = 5$.

Moreover every face has length at least 4, so

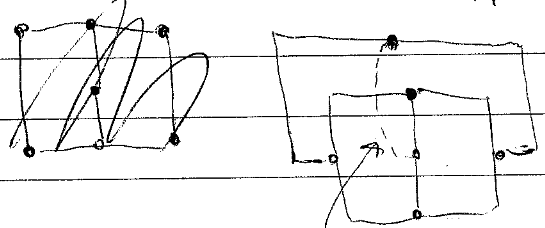
$|E(K_{3,3})| \geq \frac{5 \cdot 4}{2} = 10 > 9$, contradiction.

Claim 6: If, in a topological space Ω, a simple closed curve doesn't separate Ω, then $K_{3,3}$ embeds in Ω.

Proof 6: Because the curve is compact, it has leftmost, rightmost and topmost points.



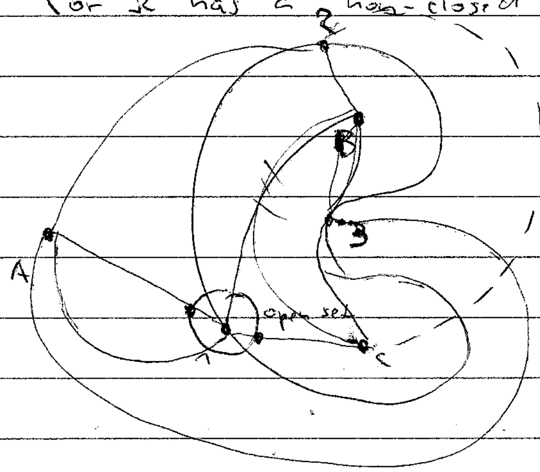
this exists by compactness of upper and lower parts of a curve



this edge exists, because curve doesn't separate

Claim 7: If, in a topological space Ω, a simple closed curve separates some 3 points from each other, then $K_{3,3}$ ~~is embeddable~~ embeds in Ω. (or Ω has a non-closed separating curve).

Proof 7:



because points are separated

A, B, C - separated points

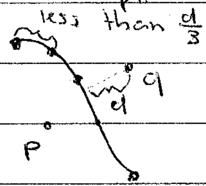
$1, 2, 3$ - arbitrary points

Then 1 2 3 — A B C is $K_{3,3}$.

We need to prove ~~claim~~ following claim to show, that three are edges $A-1$, $C-1$, ... which don't cross the curve. (only for Ω = PLANE).

Claim 8: A simple non-closed doesn't separate the plane.

Proof 8:

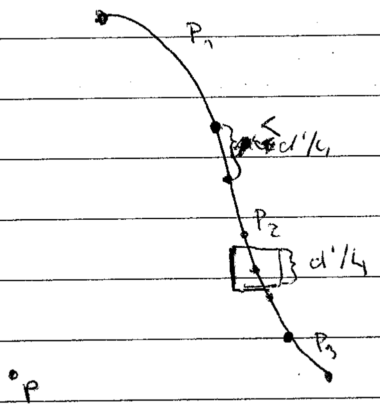


p, q are arbitrary points

Curve α, points $p, q \notin \alpha$. How to divide α into segment ~~into~~ $P_1, P_2 \dots P_k$ such as $\bigcup_{1 \le i \le k} = \alpha$.

Every two points in P_i have dist $< d/3$, where ~~dist~~ $d = \text{dist}(\{p, q\}, \alpha)$.

This follows from compactness. (of space $[0,1]$, where the curve is defined).



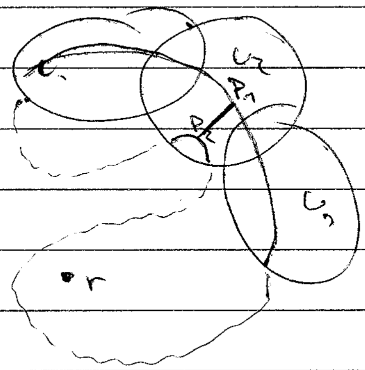
Let d' = minimum distance between non-consecutive segments.
Two consecutive segments overlap and two squares from non-following P_i, P_j, $j \geq i+2$ are disjoint.
Let U_i be the union of squares of P_i.
~~Exac~~ Each U_i is planar graph.
Formally - divide each P_i into $P_{i,1}, \dots, P_{i,l_i}$ with distances $< d'/4$ divided by points $x_{i,1} \dots x_{i,l_i+1}$ where x_{i,l_i} is the end of P_i.
- draw a square of side $d'/4$ with each center $x_{i,j}$, say $Q_{i,j}$.
- then $Q_{i,1} \cup \dots \cup Q_{i,l_i}$ is an embedding of a planar graph, say U_i.

We see ① U_i intersects U_{i-1} in ≥ 2 points.
② U_i is disjoint from U_j, $j \geq i+2$.

Now we need following claim

Claim 9: If we have a sequence of polygonal plane graphs $U_1 \dots U_k$ satisfying ① and ② and r is a point of the outer face of $U_i \cup U_{i+1}$ for each $i = 1 \dots k-1$, then r is in the outer face of $\bigcup_{1 \leq i \leq k} U_i$.

Proof 9: Assume r in the interior $U = U_1 \cup \dots \cup U_k$, then r is in the interior of some cycle C in graph U.
We choose such $C \subseteq U_i \cup \dots \cup U_j$ with minimum $j-i$.
Then $j-i > 1$.
Also minimize $|E(C) \setminus E(U_{j-1})|$.



There are two segments P_1, P_2 of C in U_{j-1}.
There is a path ~~of~~ between P_1, P_2.
We close the cycle with this path and get a cycle with ~~less~~ less $|E(C') \setminus E(U_{j-1})|$. Contradiction.