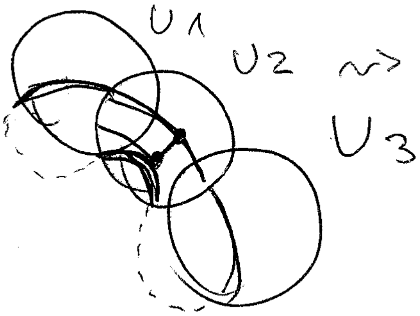
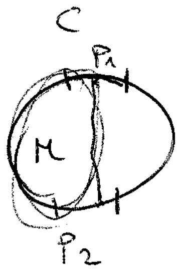
Then $j-i>1$. ~~Also~~ Also minimize $|E(G)| - |E(U_{j-1})|$ ③



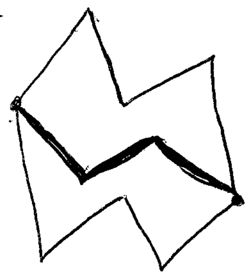
⤳


⤳ one less edge in $U_1 \cup \dots U_{j-2}$ or U_j

→ contradiction with minimum

III

Left over



Exactly 3 faces

At least 3: ≤ 3 cycles - 2 small + 1 big

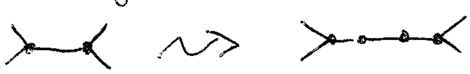
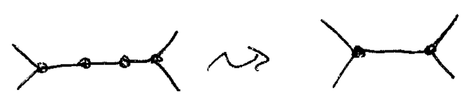
Not more than 3: if there exists 4th ~~cycle~~ face its boundary is 4th cycle but only 3 cycles.

Kuratowski thm. and comb. embeddings

lemma Every 3-connected graph on ≥ 5 vertices has an edge such that G/e is 3 connected ("contraction of e")

Proof: Assume G/e has 2-cut C: $C = \{v_e, z\}$, where v_e is the contraction $e = xy$, and so $C = \{x, y, z\}$ is a 3-cut in G. Choose e, z such that the largest component M of $G - \{x, y, z\}$ is maximized. Now contract $e' = zu$, assume G/e' has a 2-cut $\{v_{e'}, v\}$. Then $G - \{z, u, v\}$ has a component $(M \cup \{x, y\}) \setminus \{v\}$ larger ~~th~~ than M → contradiction

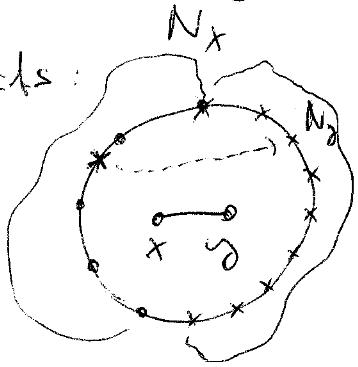
Note: In embedded graphs, vertices of degree 2 are "irrelevant":

* subdivision 
* topological minor: subgraph + 

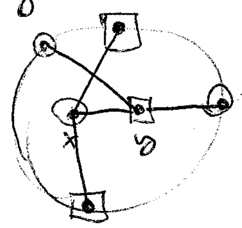
Theorem: Every 3-connected graph with ~~a~~ no $K_5, K_{3,3}$ subdivisions has a straight line convex plane embedding (edges are line segments, faces are bound by convex polygons)

$\ulcorner$ Proof: by previous lemma, for $|V(G)| \geq 5$ find e such that G/e is 3-connected and by induction, G/e has such embedding.

Possible conflicts:

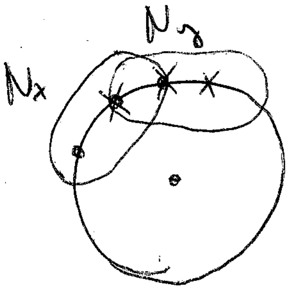


$\rightsquigarrow$



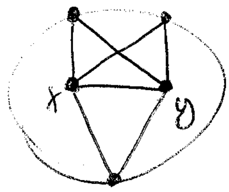
$\to K_{3,3}$

Nx - neighbourhood of x
Ny - — " — of y



2 vertices overlap and 1 additional vertex in both Nx and Ny $\to K_{3,3}$ case again

3 vertices $\to K_5$:



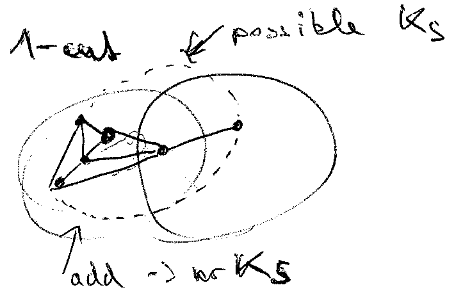
Theorem (Kuratowski) A graph has a straight-line embedding in the plane if, and only if, it has no $K_5, K_{3,3}$ subdivisions

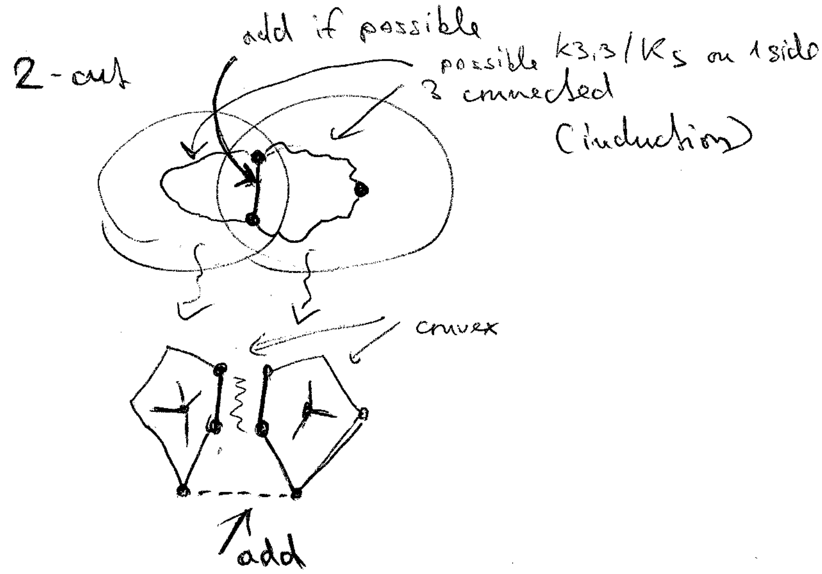
$\ulcorner$ Proof: "$\Rightarrow$": $G \geq K_5$, then K_5 has a plane embed. $\to$ topol. minor K_5 is plane. Euler: $|V| + |F| - |E| = 2$ $= 5 + |F| - 10 = 2 \Rightarrow |F| = 7 \Rightarrow$ so K_5 has $7 \cdot 3 / 2 > 10$ edges - contradiction. $K_{3,3}$ same proof.

"$\Leftarrow$": "maximise $|E(G)|$"

Claim: If adding ~~any~~ new edge to G creates a subdivision K_5 or $K_{3,3}$, then G is 3-connected. $\to$ use previous theorem
cz.

Proof of the claim:





Def: A combinatorial embedding of a graph G is a system of cyclic permutations $\Pi = (\pi_v : v \in V(G))$ where π_v is on the edges incident of v.

Def: A face in a comb. embedding Π of G is closed walk obtained as follows
- starts in $e = uv = e_1$
- for $e_i = v_{i-1}v_i$, take $e_{i+1} = \pi_{v_i}(e_i)$
- end with $e_k = e_1$ in the same direction

Let $F(G,\Pi)$ be the set of all faces

Lemma: (G,Π) connected is plane iff $|V(G)| + |F(G,\Pi)| - |E(G)| = 2$

Proof: "⇒" G is plane (embedding - picture) -> read off the cyclic permutations π_v counter clockwise. Then the formula holds by Euler

"$\Leftarrow$" (Just outline)

As long as (G,π) has >1 face, remove an edge of G shared by 2 distinct faces (then $|V|+|F|-|E| = |V| + (|F|-1) - (|E|-1) = 2$

By minimization, we assume (G,π) has one face and is connected $\Rightarrow$ $|E(G)| = |V(G)|-1$

Then G is a tree and has plane embedding any π.

$\Rightarrow$ Go back to the original graph.

Def: Geometric dual of a plane graph:

- place one vertex into every face
- draw one new edge for every edge of G between the shared faces.

Def: An abstract dual of a combinatorial embedding (G,π):

- $V(G^*) = F(G,\pi)$

  $E(G^*) \Leftrightarrow E(G)$ such that $(\varphi,\psi) \in E(G^*)$

  $\Leftrightarrow e \in E(\varphi) \cap E(\psi)$.

Def: Combinatorial dual G^* of G is such graph G^* that there is a bij. $b: E(G) \to E(G^*)$ and for every cycle $C \subseteq G$ there is a min. edge-cut $D \subseteq G^*$ such that $b(C) = D$, and vice versa.

Prop: G is planar iff it has a comb. dual. Then G^* is isom to its geometric dual.