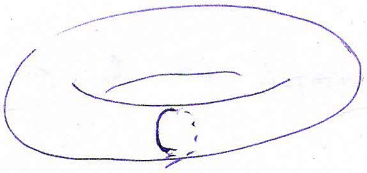
Graph Theory 27.03.14 - Cycles in embedded graphs

- This time, we shall focus on non-cellular embeddings:

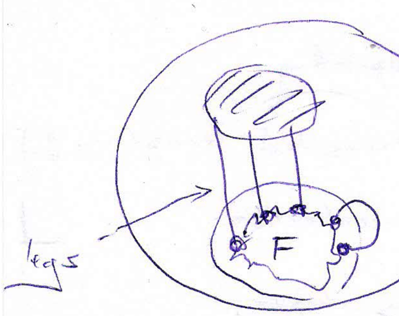


-> combinatorically $\hat{=}$ sphere

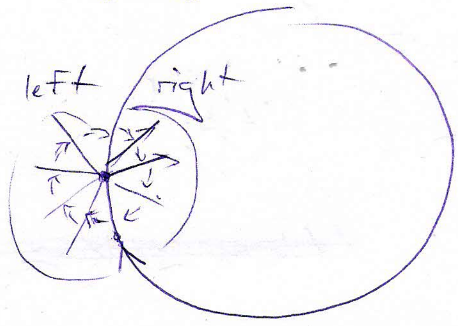
- Bridge of $F \subseteq G$: either an edge $e \notin E(F)$ but with ends in $V(F)$

  -or-

  a connected component of $G - V(F)$ plus all the incident edges of $E(G)$.



- In embedded G let $C \subseteq G$ be a cycle: the left / right edges (halfedges) of C in G are (for orientable surface C) all the edges incident with $V(C)$, not in $E(C)$ and to the left/right of C in the clockwise order around $V(C)$



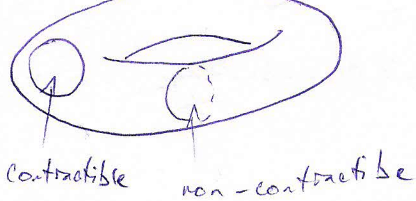
  - In case of non-orientable surface: left = right = all edges incident with $V(C)$.

- The left / right graph of C in G is the union of C and all the bridges of C containing some left / right edge of C

- A halfedge is a pair (e, v), where v is an end of e.

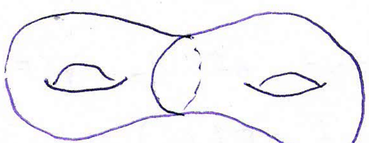
- A cycle C in an embedded graph G is (G of no loops, connected $C \neq G$)
  - two-sided if the sets of left and right (half)edges are disjoint and one-sided if the two sets coincide and are nonempty.

- contractible — if the left / right graph of C is of genus 0 (plane)



- separating — if the left and the right graph of C are disjoint up to C.

E.g.: non-contractible + separating:



⌈CLAIM: Contractible $\Rightarrow$ separating

Separating $\Rightarrow$ two-sided (except $C = G$)

One-sided $\Rightarrow$ non contractible ⌋

Every other combination may happen.

Def: Let G be a graph and K a collection of cycles of G. Then K satisfies the 3-path condition if for any $x \neq y \in V(G)$ and 3 internally-disjoint x-y paths -defining 3 cycles A, B, C, then: ~~$A \notin K \wedge B \notin K \Rightarrow C \notin K$~~

$$A, B \notin K \Rightarrow C \notin K$$ (try contracting A, B. is C contracted?)

⌈CLAIM: The following satisfies the 3-path cond:

a) K = "non contractible"

b) K = "non separating"

c) K = "one-sided"

Algorithm: Input: G and K satisfies the 3-path cond.

Output: A shortest cycle in K.

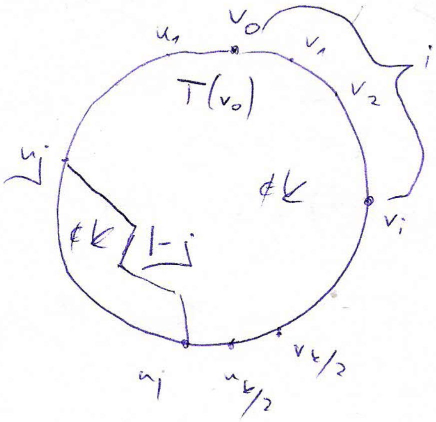
1. $\forall v \in V(G)$, construct the BFS tree T_v.
2. $\forall v \in V(G)$, $\forall e \in E(G) \setminus E(T_v)$, check the unique cycle of $T_v + e$.

Proof:

choose opt. $C \in K$

$\min |E(C) \setminus E(T_{v_0})|$

maximal ~~rez~~ ~~same~~



Distance oracles

- preprocessing + queries on shortest paths
- single-source shortest path (SSSP)
- multiple-sources " (MSSP)
- all-pairs " (APSP)

· Dijkstra (SSSP) $O(|V|^2)$, with Fibonacci's heap: $O(|E| + |V| \log |V|)$

· Fredericson (SSSP on planar graphs) $O(n\sqrt{\log n})$ (that is since $|E| \in O(n)$)

APSH $O(n^2)$

· Djidjev - ~~prep~~ $S \in [n, n^2]$

space $O(S)$, query $O(n^2/S)$

Algorithm (Klein)

MSSP on embedded graphs

- Choose a face and compute shortest paths inside.
- Create a BFS tree for one vertice.
- Reuse the tree for an adjacent vertex

$O(g\, n \log n)$

1 for planar

- Query the tree structure: $O(\log n + k \log \log \Delta)$

(another problem

Parametric SP: $\lambda \in [0, 1]$

$w_\lambda(\vec{e}) = w(\vec{e}) + \lambda \cdot w'(\vec{e})$

can change the optimal paths