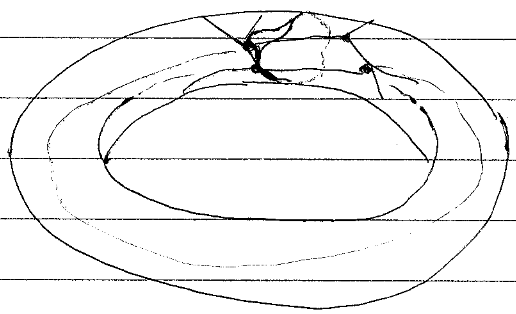
FACE-WIDTH, MINORS, KURATOWSKI

Def. The *edge-width* of embedded graph G is the length of the shortest non-cotractible cycle in G. Notation $ew(G)$. (undefined for plane).

The *dual edge-width* – the same in geometric dual

Def The *face-width* is the least number of points in which a noncontractible loop on surface intersects the embedding of G. Notation $fw(G)$.



Claim Let F_G be the vertex-face incidence graph of G. Then $fw(G) = \frac{1}{2} ew(F_G)$.

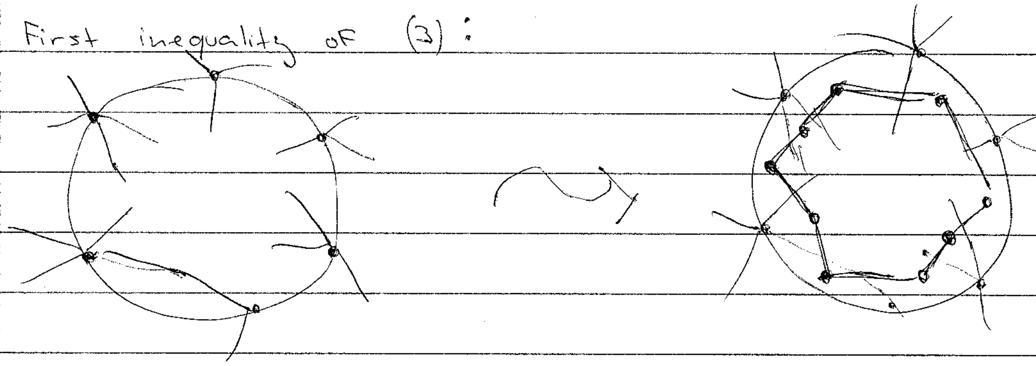
Theorem Both edge- and face-width are computable in poly. time.

Claim (1) $fw(G) = fw(G^*)$,

(2) $fw(G) \leq ew(G)$,

(3) $\frac{2}{\Delta(G)} \cdot ew(G^*) \leq fw(G) \leq ew(G^*)$

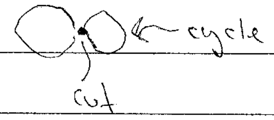
Proof First inequality of (3):



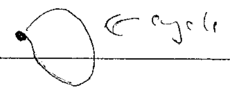
nonseparating cycle of G

Claim: All facial walks of non-planar G are cycles iff G is 2-connected and $fw(G) \geq 2$.

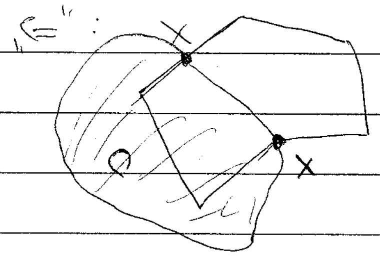
Proof: „$\Rightarrow$": If not 2-connected



If $fw(G) = 1$



„$\Leftarrow$":

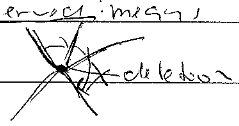


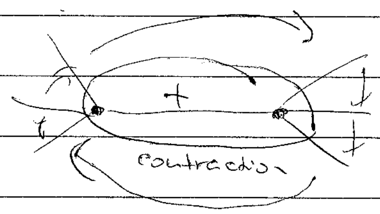
If C is planar $\rightarrow$ not 2-connected (x is cut vertex)

If both sides are not planar $\rightarrow$ C is separating

Def: G is a <u>surface-minor</u> of embedded H if G is a minor of and rotation scheme is preserved.

Where <u>preserved</u> means





So contractions can be done only on "+" edges and not on noncontractible loops

Theorem Let G be an embedded graph on surface $\Sigma \neq S_0$. Then there exists N_G such that any H emb on Σ with ~~ew~~ $fw(H) > N_G$ has G as a surface-minor.

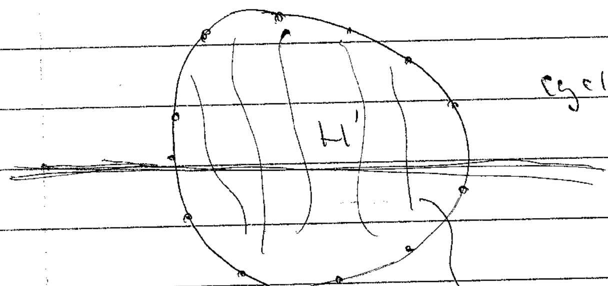
Claim Let G be plane. There exists N_G s.t. G is a surface minor of the $N_G \times N_G$ plane grid

Proof. Just take sufficiently dense grid

Claim Previous claim holds even when we fix the vertices of one face of G to the selected pairwise-distant vert. of the grid

Claim The surface-minor holds for $\Sigma = N_1$.

Proof If "$fw(H) \to \infty$" then H has a huge "grid" as a surface minor.



- There are also k horizontal disjoint paths.

So this case is converted to the planar case and can be solved by previous claim.

Theorem For every surface Σ, there is a finite set X_Σ of graphs such that H embeds in Σ iff H has no minor isomorphic to a member of X_Σ.

Proof outline
1. those in X_Σ of bounded tree-width
2. those in X_Σ not ~~of~~ of bounded tree-width have huge planar grids as minors
3. $G \in X_\Sigma \Rightarrow G - e$ embeds in Σ and a big portion of the huge grid is now plane

.... contradiction (somehow).