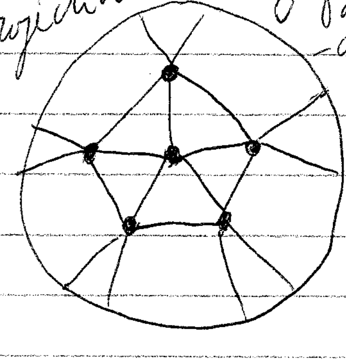
PLANAR COVERS AND EMULATORS

* Origins (Negami) – how to describe all nonequivalent embeddings of 3 – connected projective graphs (N_1).

projective embedding of K_6 on disc



* Topological "cover map" × "lift"

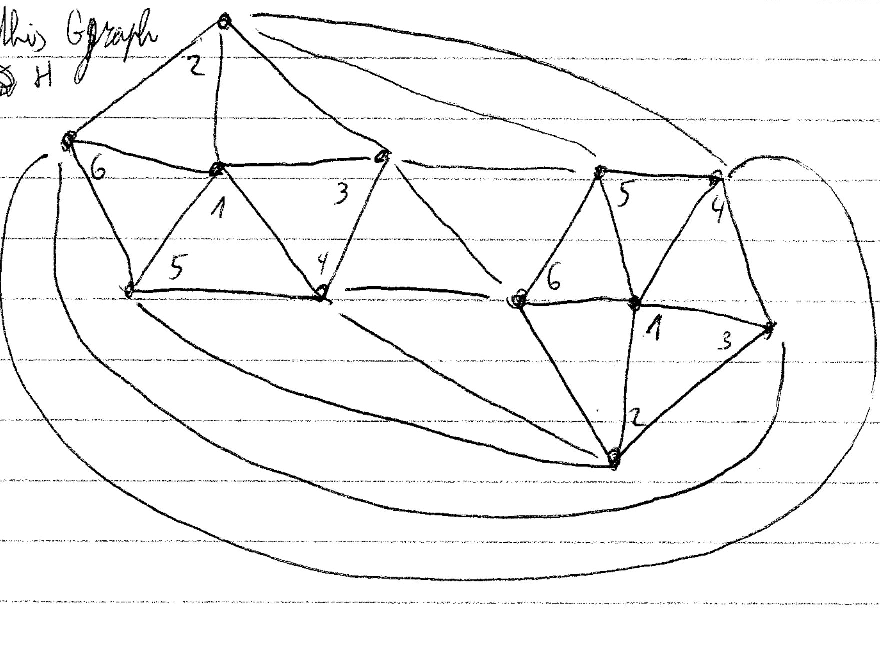
some places are "squished" into single points

inverse procedure

<u>THM</u>(Negami): The inequivalent projective embeddings of 3-con G are in a one-to-one correspondence with the <u>double planar covers</u> of G.

<u>DEF</u>: A mapping $\varphi : V(H) \to V(G)$ is a <u>cover</u> of H to G if φ is <u>onto</u> and $\forall v \in V(H)$ the neighbours of v in H are mapped bijectively onto the neighbours of $\varphi(v)$ in G.

K_6 has this ~~G~~ graph H



DEF: G has a planar cover if there exists H plane such that H covers G. (Cover is k-fold if $|V(H)| = k \cdot |V(G)|$).

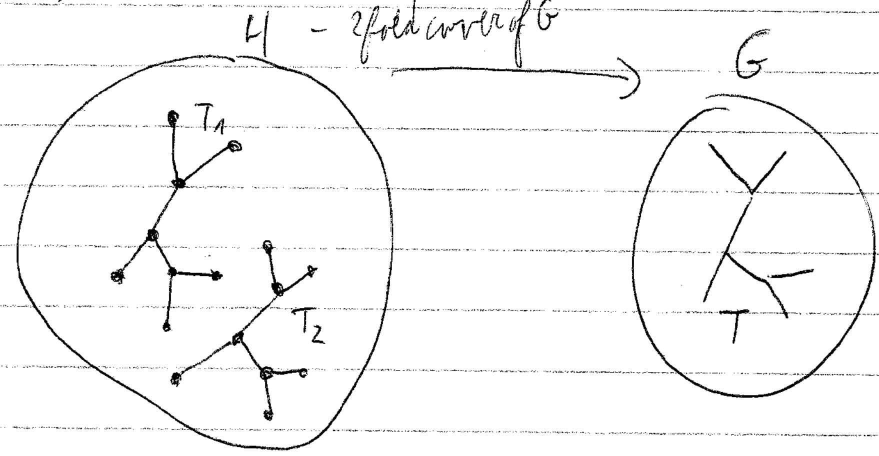
CLAIM 1: Let H cover connected G, and $T \subseteq G$ be a tree. Then the "lift" of T in H is a forest of trees isomorphic to T.

$\longrightarrow$ "k-fold" is well-defined

THM (Negami): 3-connected G has a (double) 2-fold planar cover iff G is projective.

Proof: $\Leftarrow$ lift the projective emb. of G, get the 2-fold planar cover.

$\Rightarrow$



Choose a spanning tree $T \subseteq G$, and let $T_1 \cup T_2$ be its lift in H. Then $T_1 \leftrightarrow T_2$ is an automorphism of H.
Assume now H is 3-connected.
Then for any $u_1, u_2 \in V(H)$, $\varphi(u_1) = \varphi(u_2)$, the rotation of edges around u_1 is correspondent with the rotation around u_2 (up to possible mirror): otherwise, the aut. $T_1 \leftrightarrow T_2$ of H would give a different plane embedding of H (the "same picture", though)
Then the edges "leaving" T_1 have the same cyclic order as when they are "leaving" T_2 (up to mirroring).
This gives a plane or projective embedding of G.

CONJECTURE (Negami): Connected G has a planar cover iff G is projective.