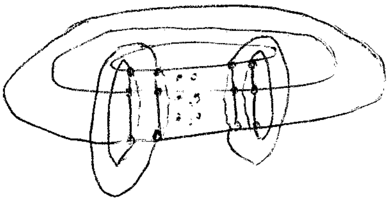
CROSSING NUMBER OF A GRAPH

Problem:

- $K_{3,n}$: $\binom{n}{3} \times K_{3,3}$; every crossing counted at most $\overset{n-2 \text{ times}}{V}$ $\Rightarrow \dfrac{\binom{n}{3}}{n-2} = \dfrac{n \cdot (n-1)}{6}$ crossings (lower bound)

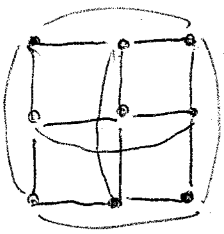
$K_{n,n}$; $n \geq 10$ open problem

- $C_n \square C_m$; $C_7 \square C_7$ unknown



$C_n \square C_3$

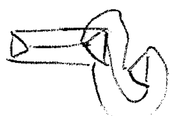
$C_3 \square C_3$



3 crossings $\rightarrow$ $C_n \square C_3$ has $\geq n$ crossings, induction from $n = 3$

triangle with crossing – remove triangle $\Rightarrow$ $n-1$ induction

no triangle with crossing $\Rightarrow$



$\Rightarrow 2n$ crossings between Δ counted 2 times

$\Rightarrow$ n crossings

Def: A drawing of G in the plane is a mapping of $V(G)$ into distinct points and ~~e~~ of $E(G)$ into simple arcs between the vert. such that.

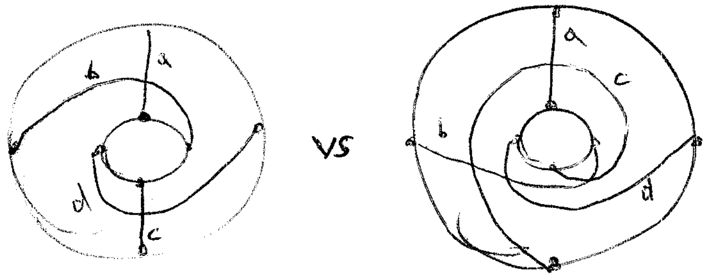
1. no three edges intersect in one point (except the end)
2. no edge contains a vertex other than its ends

Def: Crossing number $cr(G)$ is the minimum number of crossing points over all drawing of G.

Inequality of dif crossing numbers

normal vs odd

Def: Rectilinear crossing n. $\overline{cr}(G)$ is min number ... with all edges straight



ab+cd   ac+bd

+several inequalities

* $\overline{cr}(K_n)$ unknown

"$\lim_{n \to \infty}$" $\overline{cr}(K_n) \to \Theta\binom{n}{4}$

Theorem: $cr(G) \geq \frac{1}{64} \cdot \frac{|E(G)|^3}{|V(G)|^2}$ if $|E(G)| \geq 4 \cdot |V(G)|$

Proof: 1) $cr(H) \geq |E(H)| - 3|V(H)|$ by Euler

2) make H a random induced subgraph of G with probability p for each vertex; $EX(V(H)) = p \cdot n$ $(n = V(G))$

$EX(E(H)) = p^2 m$ $(m = E(G))$

$EX(\#\text{crossings in } H) = p^4 \cdot cr(G)$

$\Rightarrow p^4 \cdot cr(G) \geq p^2 m - 3pn$

$cr(G) \geq p^{-2} m - 3p^{-3} m$

choose $p = \frac{4n}{m} \leq 1$! $\Rightarrow cr(G) \geq \frac{m^3}{16n^2} - 3\frac{m^3}{64n^2}$

$= \frac{1}{64} \cdot \frac{m^3}{n^2}$

Theorem: If G is planar and $e \notin E(G)$, then computing $cr(G+e)$ is NP-hard. (Caballo + Mohar)

Theorem: If G is planar and $e \notin E(G)$, then one can in linear time approximate $cr(G+e)$ up to factor of $\Delta(G)$ $\quad (\Delta(G)/2)$

Proof: In linear time (using SPQR), one can compute ~~optimal~~ plane embedding of G in which e can be drawn with optimal number of crossings. Prove this is not worse than $\Delta(G) \cdot cr(G)$. $\square$

$e^{i\pi x}$

$U|\psi\rangle = \lambda|\psi\rangle$

Presentation: Maximum genus

genus of conn. graph $\gamma(G)$ ... smallest h that G embeds on S_h

maximum genus of connected G $\gamma_M(G)$ ... largest h, such that the graph G has a 2-cell embedding on S_h

$\xi(G,T)$ ... defeciency of a span. tree T of G = number of connected components of $G-T$ with odd number of edges

$$\xi(G) = \min_T \xi(G,T)$$

Lemma 1: e, d adjacent $\quad \xi(G-d-e,T) = 0 \Rightarrow \xi(G,T) = 0$

Lemma 2: G nontree $\quad \xi(G,T) = 0 \Rightarrow \exists\, d,e \in E(G-T): \xi(G-d-e,T) = 0$

Lemma 3: $v \in V(G)$ of least degree 3, Π one face orientable embedding $\Rightarrow \exists\, d,e \in E(G)$ adjacent : $G-d-e$ has one face orientable ~~embedding~~ embedding

Lemma 4: d,e adjacent, $G-d-e$ connected with orientable 1-face embedding $\Rightarrow G$ has one-face embedding

Lemma 5 G connected. G has one-face embedding iff $\xi(G) = 0$

1) $|E| = 0$ ✓

2) induction i) degrees 1 or 2

ii) degrees ≥ 3

Lemma 6 G connected. ~~minimum~~ $\min_{\Pi} F_G(\Pi) = \xi(G) + 1$

Equivalent to: G has orientable embedding with n+1 or fewer faces iff

$\xi(G) \leq n$