Max Genus

Antonin Klima

April 25, 2014

Surface classification

1. We have said in lecture 4, that all connected closed surfaces are homeomorphic to either:
    - S_h the sphere with h handles
        - where S_0 is the sphere, S_1 the torus, S_2 the double torus, ...
        - orientable
    - N_k the sphere with k crosscaps
        - which can be seen as a sphere with k holes each closed off by a Moebius strip along the boundary.
        - nonorientable

Genus of a graph

1. For a surface homeomorphic to S_h we define its genus to be be h.
2. For a connected graph G we define its **genus** 'gamma' $\gamma(G)$ as the smallest h such that the graph embeds in the orientable surface S_h.
   - Clearly, planar graphs embed in S_0, the sphere. So their genus is 0.
   - K_5 and $K_{3,3}$ are not planar, but embed in S_1, so their genus is 1. [example]
3. Sometimes, this is called the **minimum genus**, to emphasize the difference from the maximum genus.
4. Unfortunately, this problem is NP-complete.

Max genus of a graph

1. For a connected graph G we define its **maximum genus** $\gamma_M(G)$ as the largest h such that the graph has a 2-cell embedding on the orientable surface S_h.
   - How comes there is such an h?
2. Since we demand that the embedding is 2-cell, where each face is homeomorphic to an open disk, at least one edge goes through every handle. Otherwise we get a contradiction a face that contains a handle $\rightarrow$ it is not homeomprhic to an open disk.
3. Finding the max genus is solvable in a polynomial time. (Furst, Gross, McGeoch)

Euler's formula and its consequences

1. From lecture 4 we know the formula for Euler characteristic of orientable surface S_h:
   - $|V| - |E| + F = \chi(S_h) = 2 - 2h$
2. For a given combinatorial embedding Π, the genus h can be expessed as:
   - $h = 1 + \frac{|E|-|V|-F(\Pi)}{2}$
3. Since $f \geq 1$, we can give an easy bound for the maximum genus as:
   - $\gamma_M(G) \leq \lfloor \frac{|E|-|V|+1}{2} \rfloor$

Interpolation theorem (Duke) outline of proof

1. A connected graph G has a 2-cell embedding in S_k if and only if $\gamma(G) \leq k \leq \gamma_M(G)$.
   - Consider the combinatorial embeddings Π, Π' corresponding to the genera $\gamma(G)$ and $\gamma_M(G)$.
   - To get Π' from Π, we can modify the local rotations at each vertex one by one.
   - We can achieve this by repeatedly exchanging two consecutive edges in the local rotation at each vertex.
   - *I suppose you can imagine that from algebra. It is sort of like bubble sort.*
   - This operation can cause either 3 faces to collapse into one, one face split in 3, or no change on number of faces. [w/o proof]
   - By the Euler's formula, this changes the genus by at most one.
   - Thus, all genera between $\gamma(G)$ and $\gamma_M(G)$ are covered.

Xuong

1. For a given combinatorial embedding Π, we have the formula:
   - $h = 1 + \frac{|E|-|V|-F(\Pi)}{2}$
2. Clearly, no matter the embedding $|E|$ and $|V|$ are fixed, thus the embedding needs to minimize $F(\Pi)$ in order to maximize h and achieve $h = \gamma_M(G)$.
3. We will show that $\min_\Pi F(\Pi) = \xi(G) + 1$, where:
   - $\xi(G, T)$ is "deficiency of a spanning tree T for a connected graph G" defined as the number of connected components of G–T that have an odd number of edges.
   - $\xi(G) = \min_T \xi(G, T)$
   - example

- **Lemma 1.** Let T be a spanning tree for a connected graph G and let $d \neq e$ be adjacent edges in G–T. If furthermore $\xi(G - d - e, T) = 0$, then $\xi(G, T) = 0$.
    1. Every component adjacent to d or e in G–d–e–T has even number of edges since $\xi(G - d - e, T) = 0$.
    2. In G–T one component contains d,e and all components adjacent to d,e in G–d–e–T.
    3. The number of edges in the component of G–T that contains the edges d and e is 2+sum of all distinct components in (1).
    4. Other components in G–T remain the same, thus $\xi(G, T) = 0$. (expl)

- **Lemma 2.** Let G be a connected graph other than a tree. Let T be a spanning tree of G such that $\xi(G, T) = 0$. Then $\exists d, e \in E(G - T)$ adjacent such that $\xi(G - d - e, T) = 0$.
  1. Since G is not a tree, there is a component H in G–T with at least one edge.
  2. The component H has an even no. of edges, as $\xi(G, T) = 0$.
  3. Since H is a connected graph with at least two edges, there are adjacent edges d and e in H such that $H - d - e$ has at most one nontrivial component. Thus, $\xi(H - d - e, T) = 0$.
     - proof by induction on number of vertices
     - I.B. 3 vertices, triangle, OK.
     - I.S. Let G have n+1 vertices. Consider arbitary vertex v. By I.H., find d, e in $G - v$. If this is not appropriate, it is because v is connected with a now disconnected vertex (vertices) of G. But then we can use the edge(s) incident to v instead.

- **Lemma 3.** Let G be a connected graph such that every vertex has degree at least 3. Let G have a one-face orientable embedding Π. Then there exist adjacent edges d and e in G such that G-d-e has a one-face orientable embedding.
  1. Let d be the edge whose two occurences in the single boundary walk of the embedding Π are the closest together.
  2. Boundary walk can be written as $dAd^{-}B$, where no edge appears twice in A.
  3. A is nonempty, because G has no vertex with degree 1. We choose e as the first edge in A.
  4. Deleting d from G results in a two-face embedding. [picture]
     - By going into A and out we visited only one side of each edge in A
     - going into B visits the rest including the other sides (it can't visit sides in A else orig. walk would visit an edge twice)
  5. Deleting e from G–d results in a one-face embedding. [picture]

- **Lemma 4.** Let $d, e \in E(G)$ adjacent edges of a connected graph G such that $G - d - e$ is a connected graph having an orientable one-face embedding. Then the graph G has a one-face orientable embedding.

  1. Let $d = uv$, $e = vw$
  2. Extend the two-face embedding $(G - d - e) \to S$ to a two-face embedding $(G - e) \to S$ by placing the image of d across the single face.
  3. Attach a handle from one face of $(G - e)$ to the other and place the edge e so that it runs across the handle $\to$ obtain one face embedding.

Xuong 5 – 3.4.12

- **Lemma 5.** Let G be a connected graph. Then G has a one-face orientable embedding if and only if $\xi(G) = 0$.
- proof by induction on $|E|$.
  1. $|E|$=0. Trivially true.
  2. $|E| = n \rightarrow |E| = n + 1$.
- G has a vertex v of degree 1 or 2
  - Contract an edge e incident on vertex v. Denote resulting graph G'.
  - G has one-face orientable embedding if and only if G' does.
  - $\xi(G') = 0$ if and only if $\xi(G) = 0$.
  - G' has n edges, thus by I.H, q.e.d.

Xuong 6 – 3.4.12 cont'd

- Each vertex of G has degree at least 3.
  1. "$\Rightarrow$" Suppose G has a one-face orientable embedding.
     - By lemma 3 there exist adjacent edges $d \neq e$ in G such that G–d–e has a one-face orientable embedding.
     - By I.H. $\xi(G - d - e) = 0$, so there exists a spanning tree T of G–d–e such that $\xi(G - d - e, T) = 0$.
     - T also spans G.
     - By lemma 1 $\xi(G, T) = 0$, which implies $\xi(G) = 0$.
  2. "$\Leftarrow$" Assume $\xi(G) = 0$.
     - There exists a spanning tree T of G such that $\xi(G, T) = 0$.
     - By lemma 2 there exist adjacent edges $d \neq e$ such that $\xi(G - d - e) = 0$. By I.H. G–d–e has a one-face orientable embedding. By lemma 4 , so does G.

- **Lemma 6.** Let G be a connected graph. Then the minimum number of faces in any orientable imbedding of G is exactly $\xi(G) + 1$. (and this minimum is achieved by some embedding)
  1. Prove an equivalent statement
     - The graph G has an orientable imbedding with n+1 or fewer faces if and only if $\xi(G) \leq n$.
  2. By induction on n.
  3. I.B. for $n = 0$ is proven by lemma 5.
  4. Suppose theorem holds for all $k \leq n \to n$
     - "$\Rightarrow$" Suppose Π is an orientable embedding with $|F| = n + 1$.
     - There exists an edge e common to two distinct faces. (otherwise there is only one face)
     - Delete this edge $\to$ the two faces become one, resulting embedding has n faces. By I.H. $\xi(G - e) \leq n - 1$.
     - Since $\xi(G - e) \leq n - 1$, there exists T such that $\xi(G - e, T) \leq n - 1$. But T is also a spanning tree of G and $\xi(G, T) \leq \xi(G - e, T) + 1 \leq n$. Thus, $\xi(G) \leq n$.

Xuong 7 – 3.4.13

- **Lemma 6. – cont'd** Let G be a connected graph. Then the minimum number of faces in any orientable imbedding of G is exactly $\xi(G) + 1$. (and this minimum is achieved by some embedding)
  1. Prove an equivalent statement
     - The graph G has an orientable imbedding with n+1 or fewer faces if and only if $\xi(G) \leq n$.
  2. Suppose theorem holds for all $k \leq n \rightarrow n$
     - "$\Leftarrow$" Suppose $\xi(G) = n$.
     - There is a spanning tree T of G such that $\xi(G, T) = n$
     - Let H be a component of G–T with an odd number of edges.
     - Either there exists an edge e in H such that removing this edge doesn't disconnect H $\rightarrow \xi(G - e, T) = n - 1$ as this makes H have even number of edges.
     - Or there H is a tree and there is a leaf that we can disconnect. Again $\rightarrow \xi(G - e, T) = n - 1$.
     - By I.H. $G - e$ has an orientable embedding with at most n faces. Therefore G has an orientable embedding with at most n+1 faces.

Xuong theorem

- **Corollary (Xuong)** Let G be a connected graph. Then $\gamma_M(G) = \frac{|E|-|V|-\xi(G)+1}{2} = \frac{1}{2}(\beta(G) + \xi(G))$.
- Where $\beta(G) = |E| - |V| + 1$ is the first Betti number of G.
  - $2 - 2h = |V| - |E| + (\xi(G) + 1)$
  - Altough it looks like we have only proven $\geq$, assumption that $\min_\Pi F(\Pi) < \xi(G) + 1$ leads to a contradiction.