Digital Signal Processing

Periodic Sampling

Moslem Amiri, Václav Přenosil
Masaryk University

Periodic Sampling

- Periodic sampling
  - Process of representing a continuous signal with a sequence of discrete data values
  - In practice, sampling is performed by applying a continuous signal to an analog-to-digital (A/D) converter
  - Primary concern is how fast a given continuous signal must be sampled to preserve its information content

Aliasing

- Example: given following sequence of values

$$x(0) = 0, x(1) = 0.866, x(2) = 0.866, x(3) = 0, x(4) = -0.866, x(5) = -0.866, x(6) = 0$$

  - They represent values of a time-domain sinewave taken at periodic intervals
  - Draw that sinewave

Aliasing

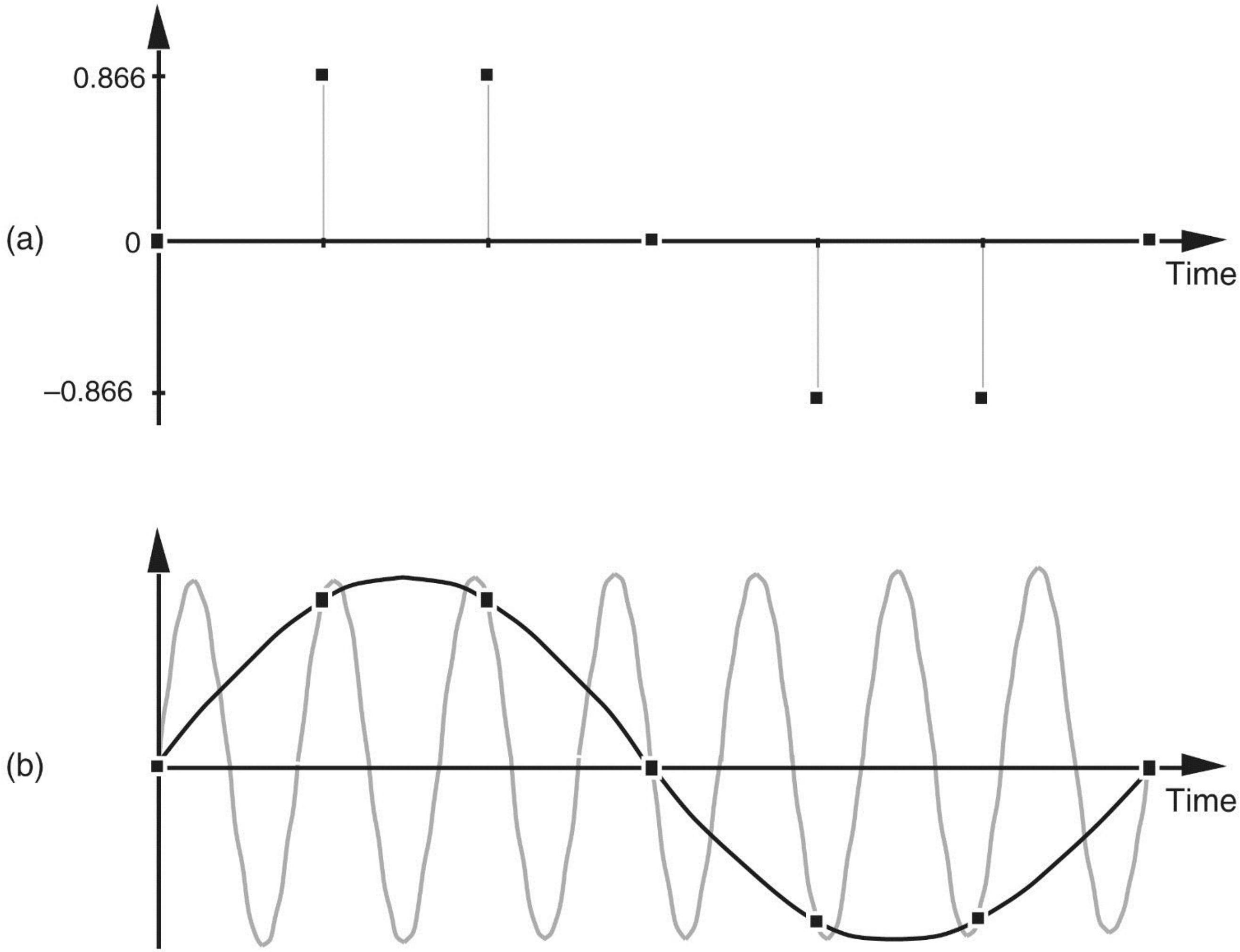


Figure 2–1 Frequency ambiguity: (a) discrete-time sequence of values; (b) two different sinewaves that pass through the points of the discrete sequence.

Aliasing

- Frequency ambiguity
  - If data sequence represents periodic samples of a sinewave, we cannot unambiguously determine frequency of sinewave from those sample values alone

Aliasing

- Mathematical origin of frequency ambiguity

$$x(t) = \sin(2\pi f_o t)$$

$$x(n) = \sin(2\pi f_o n t_s) = \sin(2\pi f_o n t_s + 2\pi m) = \sin(2\pi (f_o + \frac{m}{n t_s}) n t_s)$$

$$\xrightarrow{\text{if } m = kn} x(n) = \sin(2\pi (f_o + \frac{k}{t_s}) n t_s)$$

$$\xrightarrow{f_s = 1/t_s} x(n) = \sin(2\pi f_o n t_s) = \sin(2\pi (f_o + k f_s) n t_s)$$

  - **When sampling at a rate of f_s samples/second, if k is any positive or negative integer, we cannot distinguish between sampled values of a sinewave of f_o Hz and a sinewave of $(f_o + k f_s)$ Hz**

Aliasing

- Frequency ambiguity (*aliasing*) effects
  - Spectrum of any discrete series of sampled values contains periodic replications of original continuous spectrum
  - Period between these replicated spectra in frequency domain is always f_s
  - Spectral replications repeat all the way in both directions of frequency spectrum

Aliasing

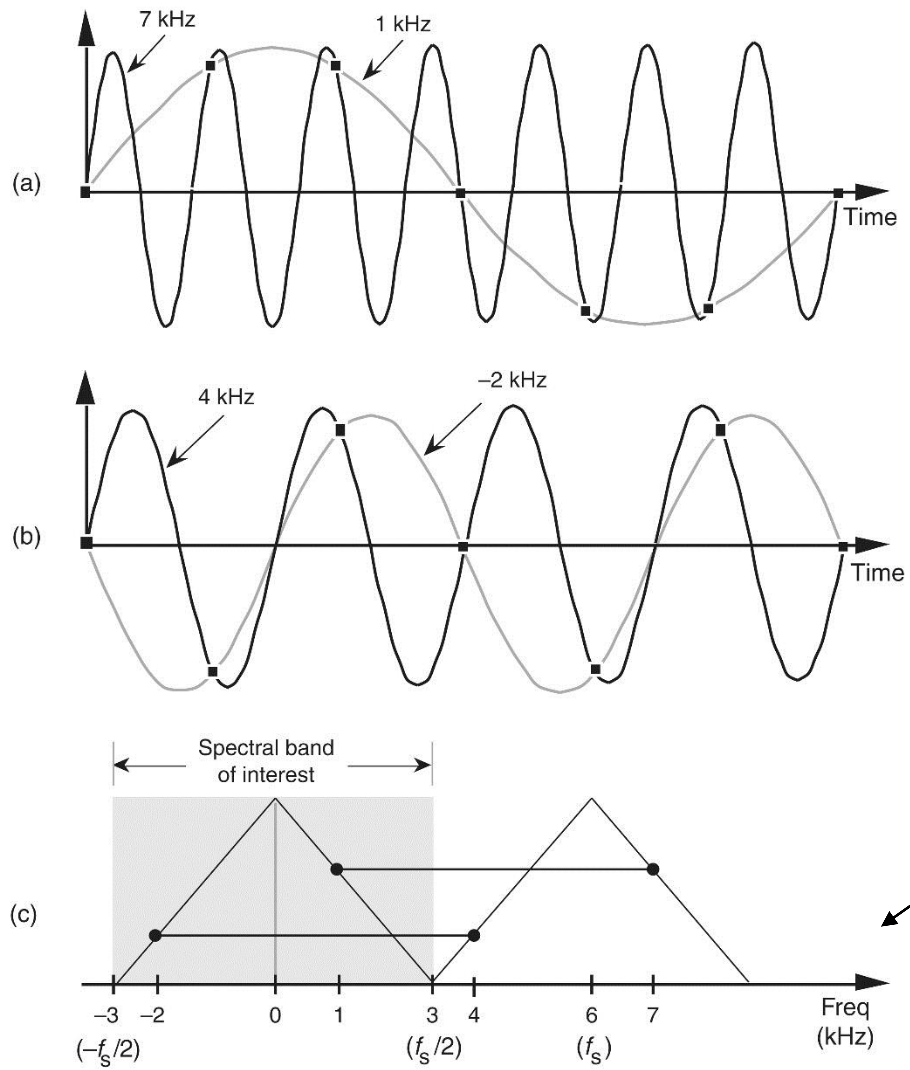


Figure 2-2 Frequency ambiguity effects of Eq. (2-5): (a) sampling a 7 kHz sinewave at a sample rate of 6 kHz; (b) sampling a 4 kHz sinewave at a sample rate of 6 kHz; (c) spectral relationships showing aliasing of the 7 and 4 kHz sinewaves.

f_o = 7 kHz, f_s = 6 kHz
k = −1 → f_o+kf_s = [7+(−1·6)] = 1 kHz
No processing scheme can determine if sequence of sampled values came from a 7 kHz or a 1 kHz sinusoid
1 kHz is an *alias* of 7 kHz

f_o = 4 kHz, f_s = 6 kHz
k = −1 → f_o+kf_s = [4+(−1·6)] = −2 kHz

$f_s/2$ is an important quantity, referred to by critical Nyquist, half Nyquist, or folding frequency

we're interested in signal components that are aliased into frequency band between $-f_s/2$ and $+f_s/2$

Aliasing

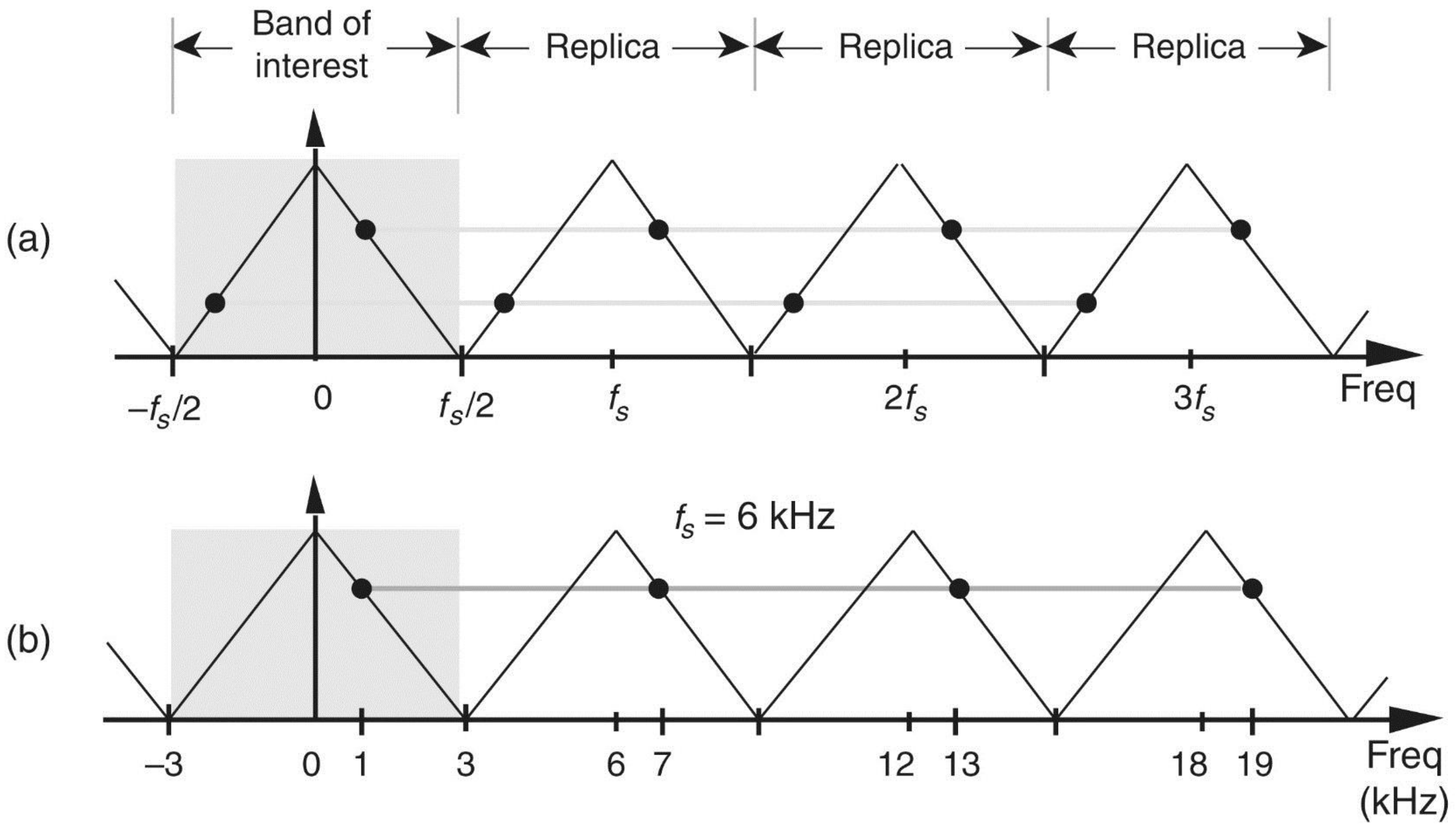


Figure 2–3 Shark's tooth pattern: (a) aliasing at multiples of the sampling frequency; (b) aliasing of the 7 kHz sinewave to 1 kHz, 13 kHz, and 19 kHz.

Sampling Lowpass Signals

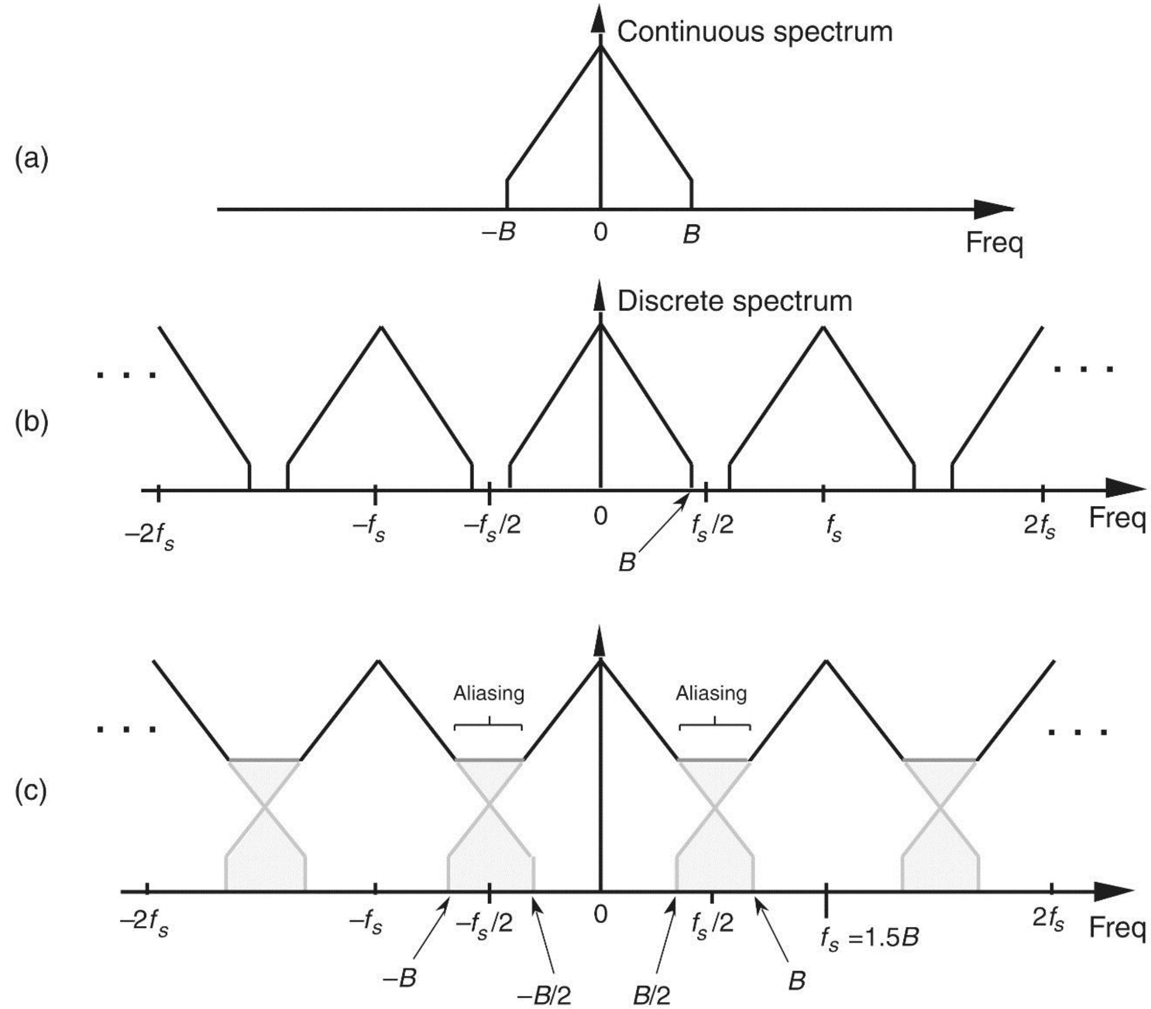


Figure 2–4 Spectral replications: (a) original continuous lowpass signal spectrum; (b) spectral replications of the sampled lowpass signal when $f_s/2 > B$; (c) frequency overlap and aliasing when the sampling rate is too low because $f_s/2 < B$.

Sampling Lowpass Signals

- Fig. 2-4(a)
  - Spectrum of a continuous real-valued lowpass $x(t)$ signal
  - Spectrum is symmetrical around zero Hz
  - Signal is *band-limited*
    - Its spectral amplitude is zero above $+B$ Hz and below $-B$ Hz
  - $x(t)$ time signal is called a *lowpass signal* because its spectral energy is low in frequency
  - Spectrum of a continuous signal *cannot* be represented in a digital machine in its current band-limited form → replicated form of (b)

Sampling Lowpass Signals

- Nyquist criterion
  - $f_s \geq 2B$, to separate spectral replications at *folding frequencies* of $\pm f_s/2$
- Fig. 2-4(c)
  - Sampling frequency is lowered to $f_s = 1.5B$ Hz
  - Lower edge and upper edge of spectral replications centered at $+f_s$ and $-f_s$ now lie in band of interest
    - Equivalent to original spectrum folding to left at $+f_s/2$ and folding to right at $-f_s/2$
    - Spectral information in bands of $-B$ to $-B/2$ and $B/2$ to B Hz is corrupted (aliasing errors)

Sampling Lowpass Signals

- A key property of band $\pm f_s/2$ Hz
  - Entire spectral content (any signal energy located above $+B$ Hz and below $-B$ Hz) of original continuous spectrum always ends up in band of interest between $-f_s/2$ and $+f_s/2$ after sampling, regardless of sample rate
  - For this reason, continuous (analog) *lowpass* filters are necessary in practice

Sampling Lowpass Signals

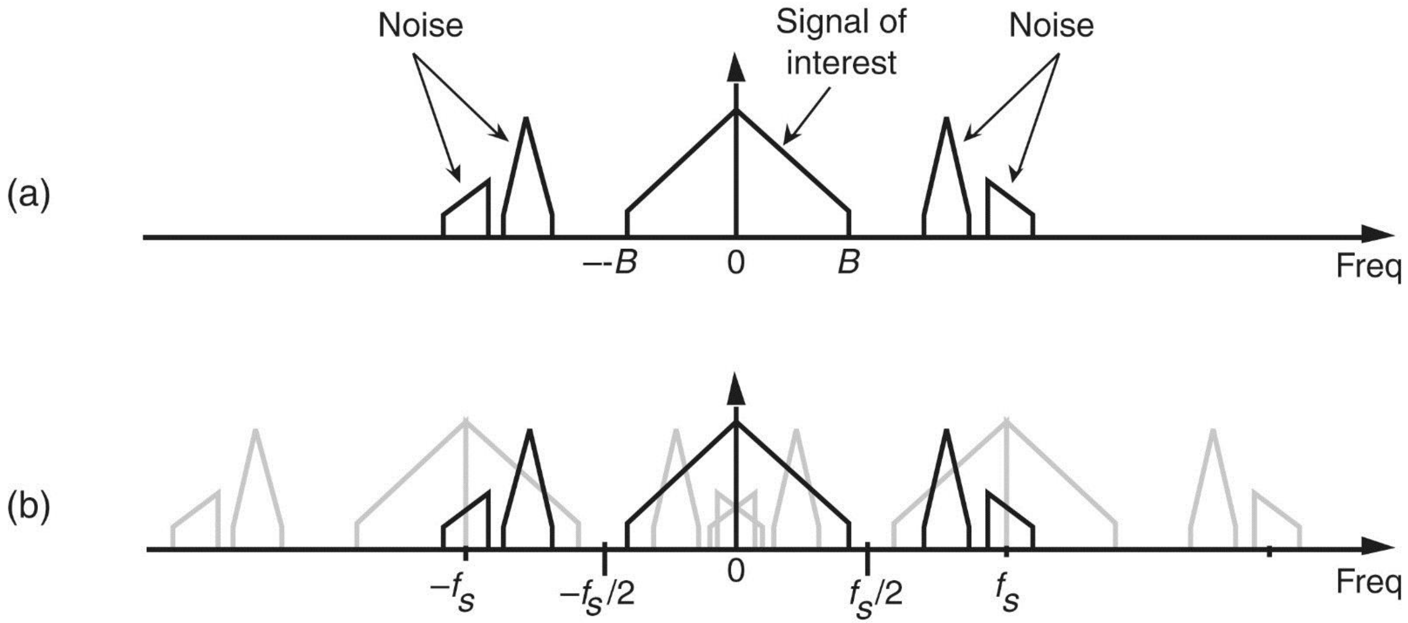


Figure 2–5 Spectral replications: (a) original continuous signal-plus-noise spectrum; (b) discrete spectrum with noise contaminating the signal of interest.

Sampling Lowpass Signals

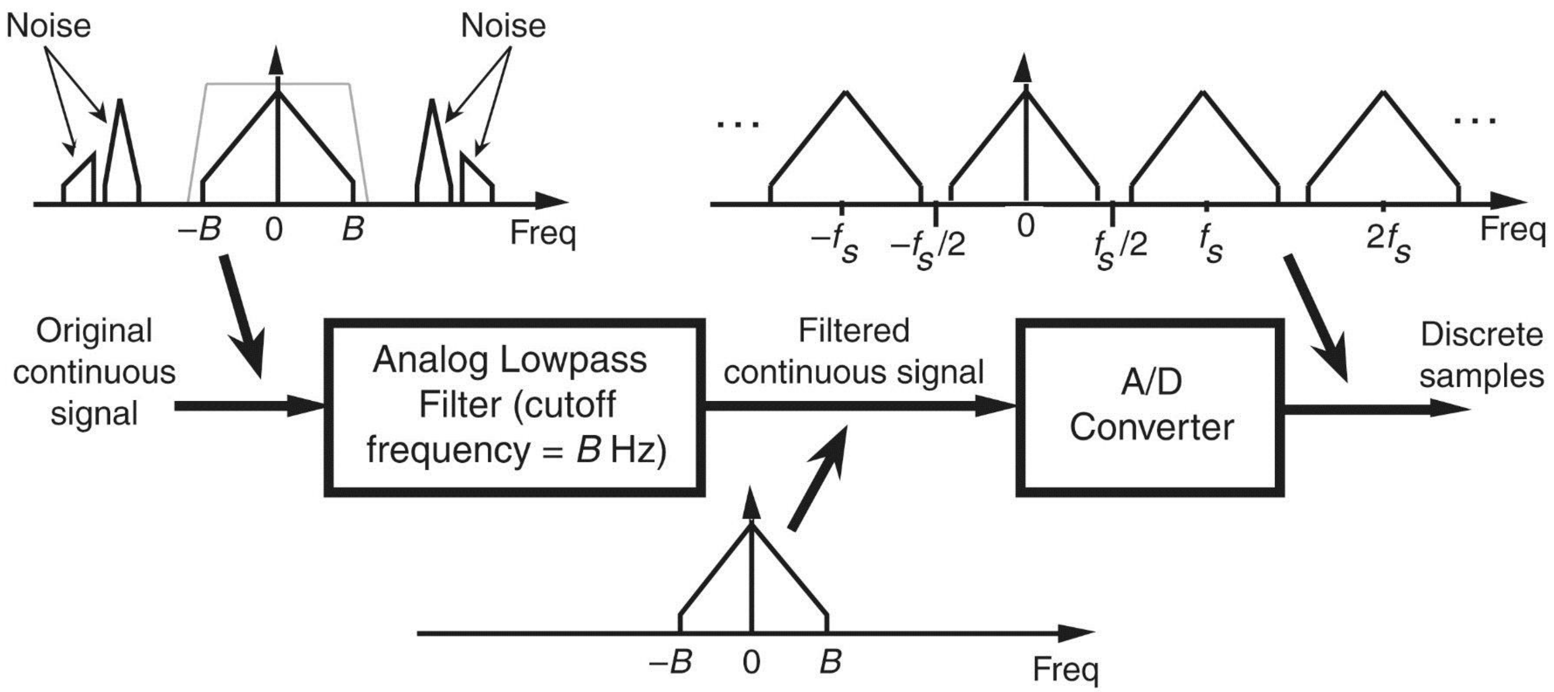


Figure 2–6 Lowpass analog filtering prior to sampling at a rate of f_s Hz.

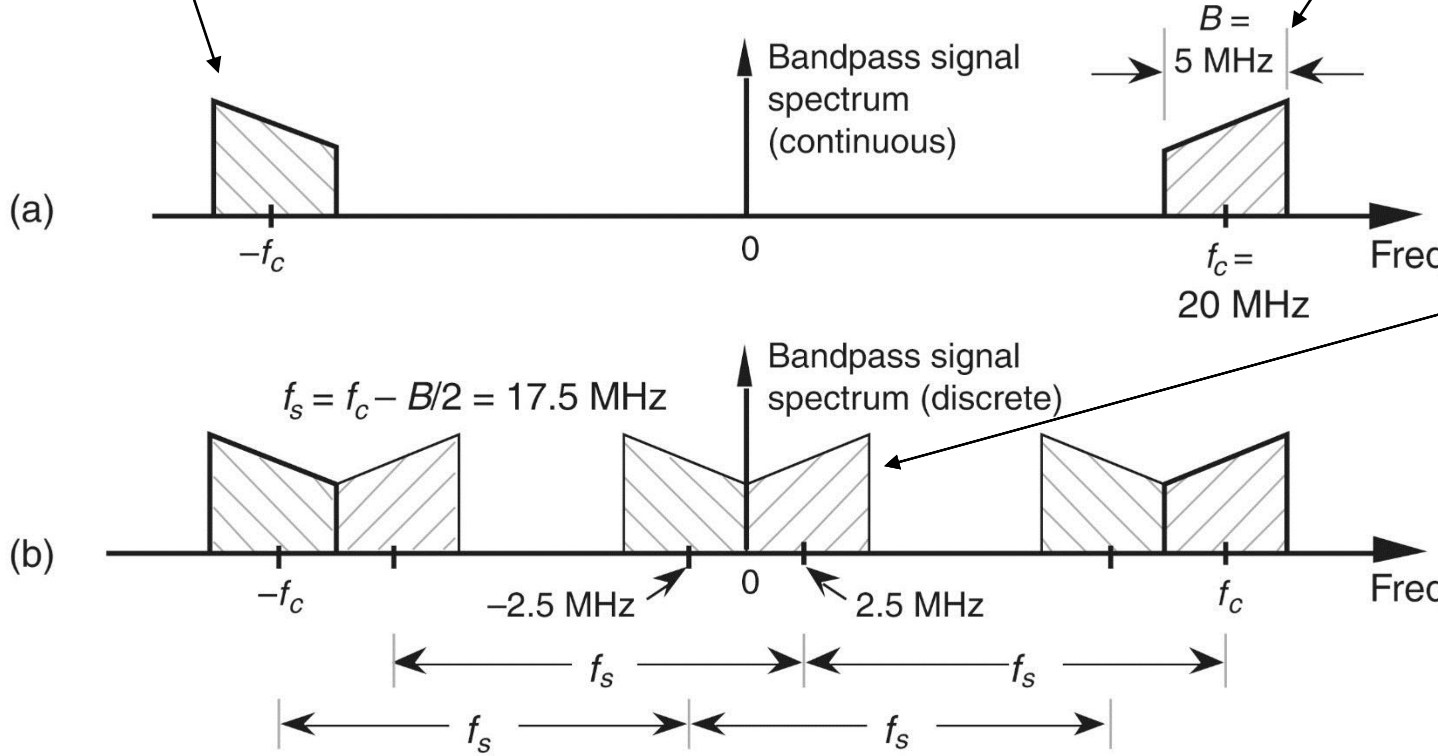
Sampling Bandpass Signals

- Bandpass sampling
  - A technique to sample a continuous bandpass signal that is centered about some frequency other than zero Hz
  - Reduces speed requirement of A/D converters below that necessary with traditional lowpass sampling
  - Reduces amount of digital memory necessary to capture a given time interval of a continuous signal
  - We’re more concerned with a signal’s bandwidth than its highest-frequency component

Sampling Bandpass Signals

negative frequency portion of signal is mirror image of positive frequency portion (real signal)

highest-frequency = 22.5 MHz
Nyquist criterion → sampling frequency must be a minimum of 45 MHz



if f_s = 17.5 MHz, spectral replications are located exactly at baseband

Figure 2-7 Bandpass signal sampling: (a) original continuous signal spectrum; (b) sampled signal spectrum replications when sample rate is 17.5 MHz.

sampling at 45 MHz was unnecessary to avoid aliasing—instead we've used spectral replicating effects to our advantage

Sampling Bandpass Signals

- Sampling translation
  - Bandpass sampling performs digitization and frequency translation in a single process
- We can sample at some still lower rate and avoid aliasing

Sampling Bandpass Signals

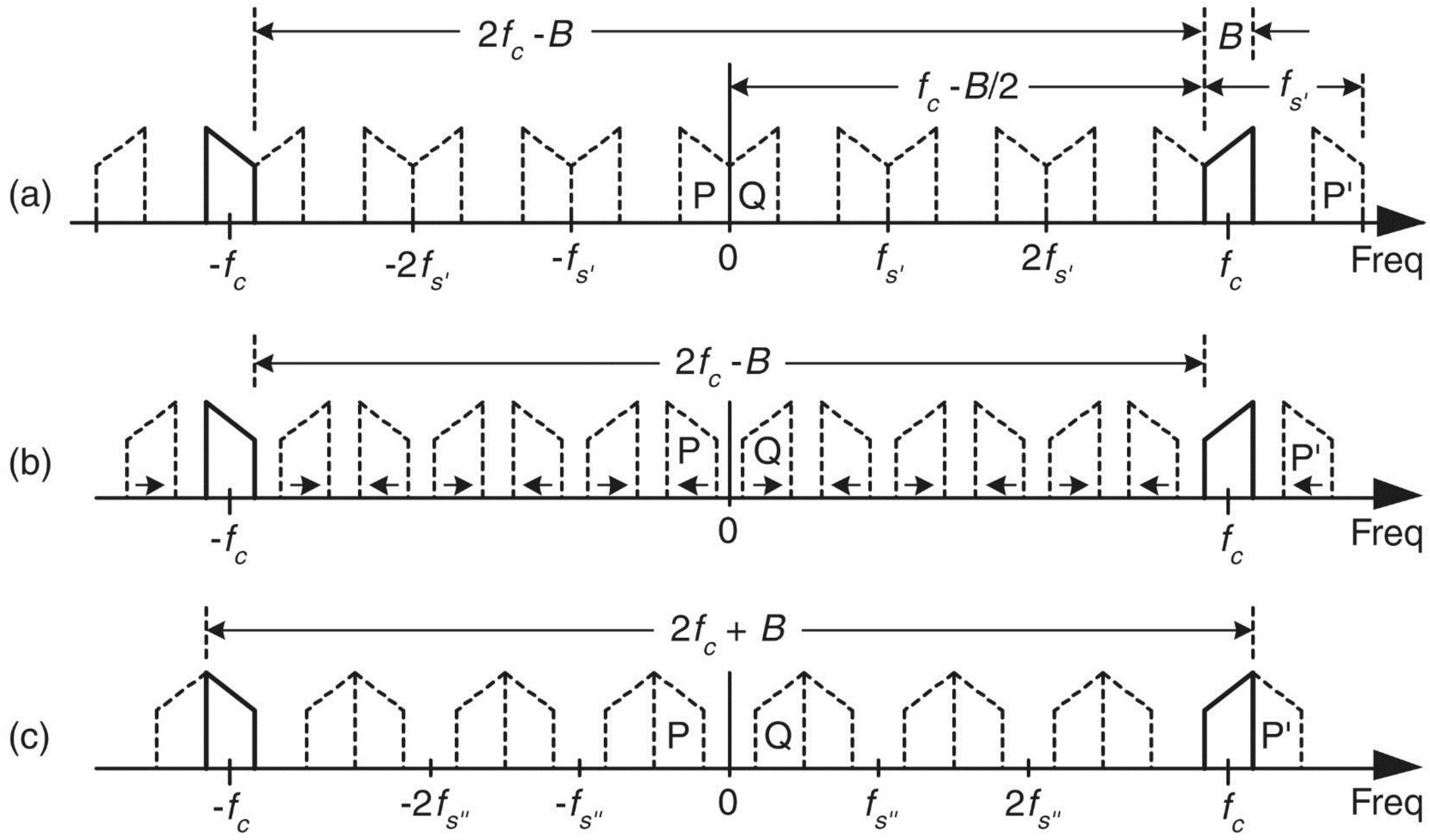


Figure 2–8 Bandpass sampling frequency limits: (a) sample rate $f_{s'} = (2f_C - B)/6$; (b) sample rate is less than $f_{s'}$; (c) minimum sample rate $f_{s''} < f_{s'}$.

Sampling Bandpass Signals

- Fig. 2-8(a)
  - Continuous input bandpass signal of bandwidth *B*
  - *Carrier frequency* (signal is centered at) = f_c Hz
  - Sample rate = $f_{s'}$ Hz → spectral replications of positive and negative bands, Q and P, butt up against each other at zero Hz

$$mf_{s'} = 2f_c - B \quad \text{or} \quad f_{s'} = \frac{2f_c - B}{m}$$

  - *m* = an arbitrary number of replications in the range of $2f_c - B$
    - *m* can be any positive integer so long as $f_{s'}$ is never less than 2*B*

Sampling Bandpass Signals

- Fig. 2-8
  - If $f_{s'}$ is increased, original spectra (bold) do not shift, but all replications will shift
  - At zero Hz, P band shifts to right, and Q band shifts to left
  - These replications will overlap and aliasing occurs
  - Thus, for an arbitrary *m,* there is a frequency that sample rate must not exceed

$$f_{s'} \leq \frac{2f_c - B}{m}$$

Sampling Bandpass Signals

- Fig. 2-8(b) and (c)
  - If we reduce sample rate below $f_{s'}$ shown in (a), spacing between replications will decrease in direction of arrows in (b)
    - Original spectra do not shift
  - At some sample rate $f_{s''}$ ($f_{s''} < f_{s'}$), replication P′ will butt up against positive original spectrum at f_c as shown in (c)

$$(m+1) f_{s''} = 2 f_c + B \quad \text{or} \quad f_{s''} = \frac{2 f_c + B}{m+1}$$

  - $f_{s''}$ decreased → aliasing occurs

$$f_{s''} \geq \frac{2 f_c + B}{m+1}$$

Sampling Bandpass Signals

- To avoid aliasing, f_s may be chosen anywhere in the range

$$\frac{2f_c - B}{m} \geq f_s \geq \frac{2f_c + B}{m+1} \qquad (1)$$

  - m is an arbitrary, positive integer ensuring $f_s \geq 2B$

Sampling Bandpass Signals

- Example (Fig. 2-7(a))
  - f_c = 20 MHz, B = 5 MHz

m	$(2f_c\text{-}B)$ / m	$(2f_c\text{+}B)$ / (m+1)	Optimum sampling rate
1	35.0 MHz	22.5 MHz	22.5 MHz
2	17.5 MHz	15.0 MHz	17.5 MHz
3	11.66 MHz	11.25 MHz	11.25 MHz
4	8.75 MHz	9.0 MHz	---
5	7.0 MHz	7.5 MHz	---

  - Sample rates below 11.25 MHz unacceptable
    - Will not satisfy Eq. (1) as well as $f_s \geq 2B$
  - Optimum sampling frequency is the frequency where spectral replications butt up against each other at zero Hz

Sampling Bandpass Signals

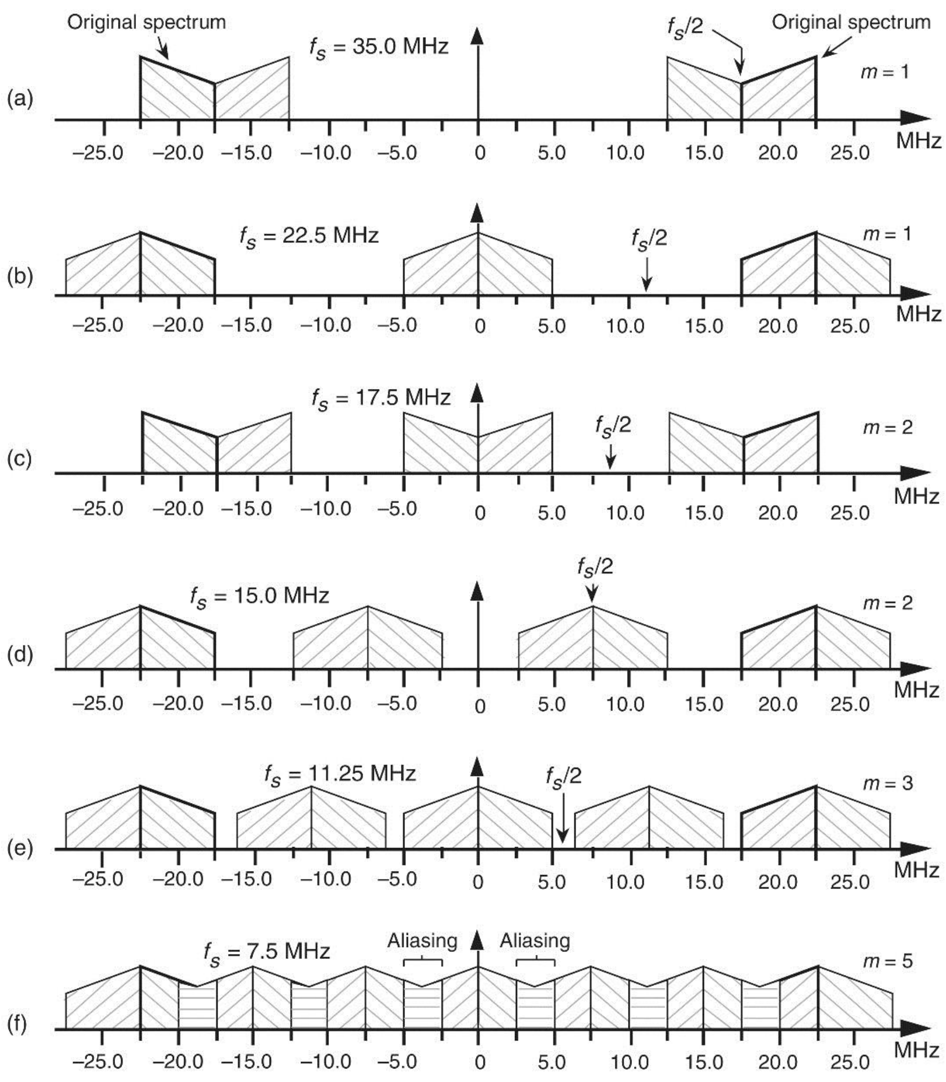


Figure 2-9 Various spectral replications from Table 2-1: (a) f_s = 35 MHz; (b) f_s = 22.5 MHz; (c) f_s = 17.5 MHz; (d) f_s = 15 MHz; (e) f_s = 11.25 MHz; (f) f_s = 7.5 MHz.

Practical Aspects of Bandpass Sampling

- Spectral Inversion in Bandpass Sampling
  - Some of permissible f_s values from Eq. (1) provide a sampled baseband spectrum (located near zero Hz) that is inverted from original analog signal's positive and negative spectral shapes
  - Happens when *m*, in Eq. (1), is an odd integer
    - We can invert spectrum back to its original orientation
    - Discrete spectrum of any digital signal can be inverted by multiplying signal's discrete-time samples by $(-1)^n$
    - Center of flipping is $f_s/4$ Hz (and $-f_s/4$ Hz)
  - When original positive spectral bandpass components are symmetrical about f_c frequency, spectral inversion presents no problem

Practical Aspects of Bandpass Sampling

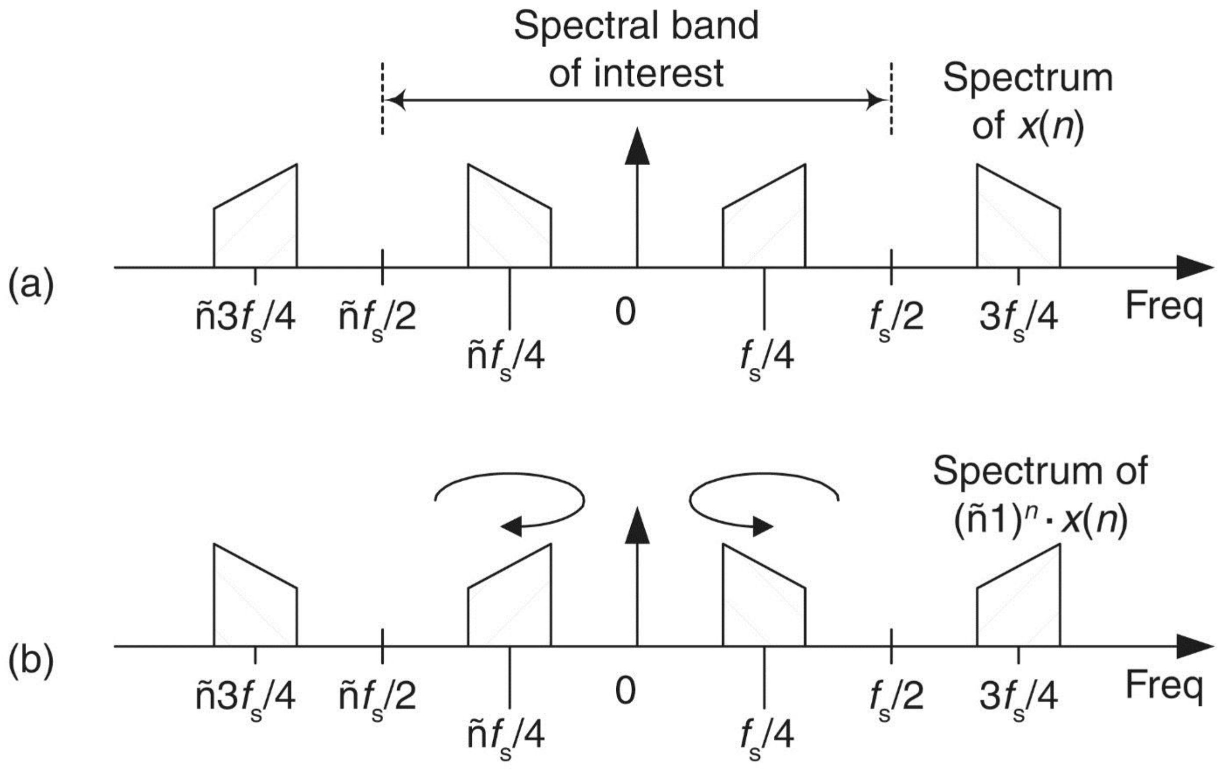


Figure 2–10 Spectral inversion through multiplication by $(-1)^n$: (a) spectrum of original $x(n)$; (b) spectrum of $(-1)^n \cdot x(n)$.

Practical Aspects of Bandpass Sampling

- Positioning sampled spectra at $f_s/4$
  - In many signal processing applications it is useful to use an f_s bandpass sampling rate that forces sampled spectra to be centered exactly at $\pm f_s/4$
  - To ensure that sampled spectra reside at $\pm f_s/4$, select f_s using

$$f_s = \frac{4f_c}{2k-1}, \text{ where } k = 1,2,3,\ldots$$

Practical Aspects of Bandpass Sampling

- Noise in bandpass-sampled signals
  - Signal-to-noise ratio (SNR) is ratio of power of a signal over total background noise power
  - Negative aspect of bandpass sampling
    - SNR of digitized signal is degraded
    - All of background spectral noise (Fig. 2-11(b)) resides in range of $-f_s/2$ to $f_s/2$ (Fig. 2-11(c))
    - Bandpass-sampled background noise power increases by a factor of $m + 1$ (denominator of right-side ratio in Eq. (1)) while signal power P remains unchanged
    - Bandpass-sampled signal's SNR is reduced by

$$D_{SNR} = 10 \cdot \log_{10}(m+1)\ dB$$

below SNR of original analog signal

Practical Aspects of Bandpass Sampling

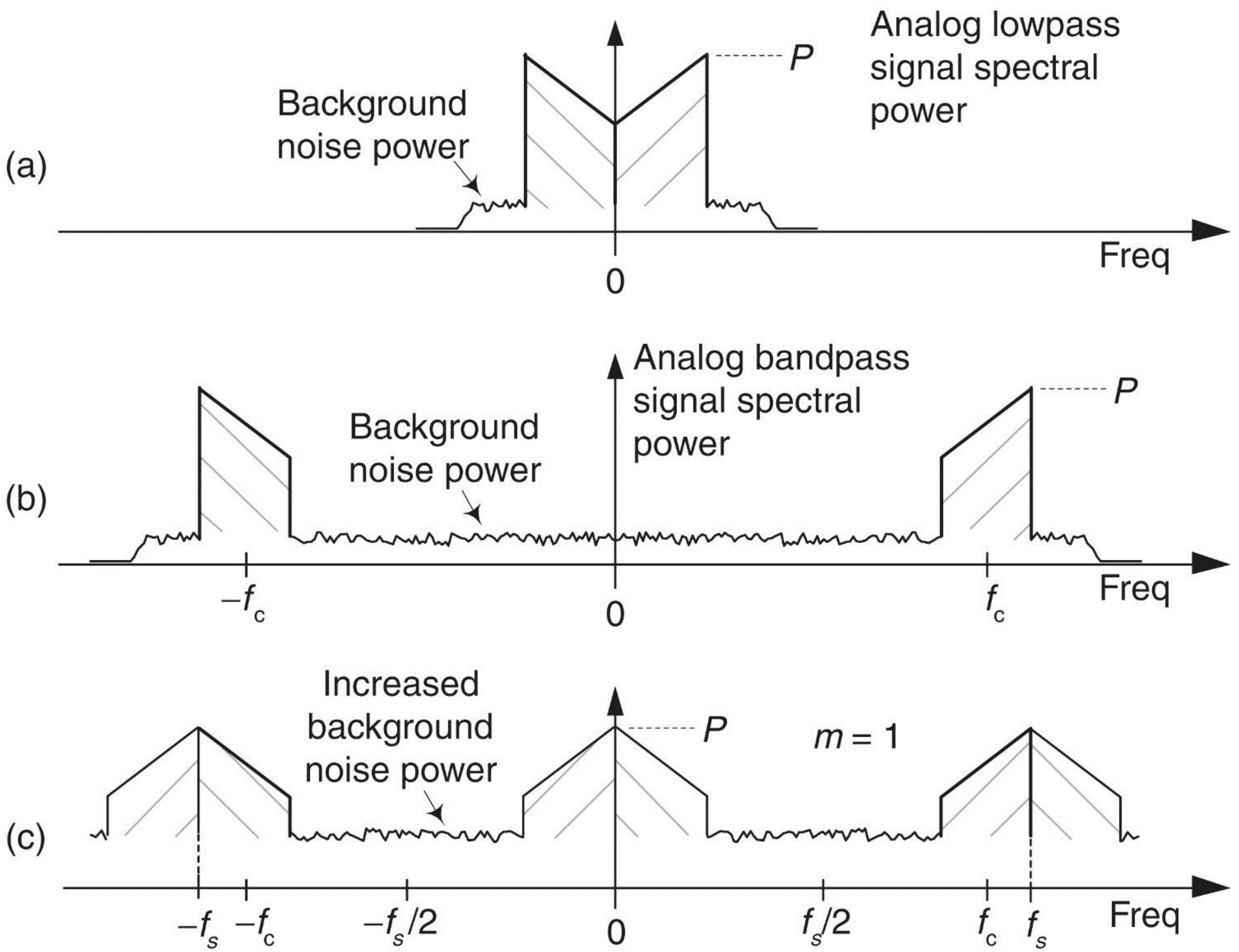


Figure 2–11 Sampling SNR degradation: (a) analog lowpass signal spectral power; (b) analog bandpass signal spectral power; (c) bandpass-sampled signal spectral power when $m = 1$.