Digital Signal Processing

The Discrete Fourier Transform (2)

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- Zero padding
  - A method to improve DFT spectral estimation
  - Involves addition of zero-valued data samples to an original DFT input sequence to increase total number of input data samples
  - Investigating zero-padding technique illustrates DFT's property of frequency-domain sampling
    - When we sample a continuous time-domain function, having a CFT, and take DFT of those samples, the DFT results in a frequency-domain sampled approximation of the CFT
    - The more points in DFT, the better DFT output approximates CFT

DFT Resolution, Zero Padding, Frequency-Domain Sampling

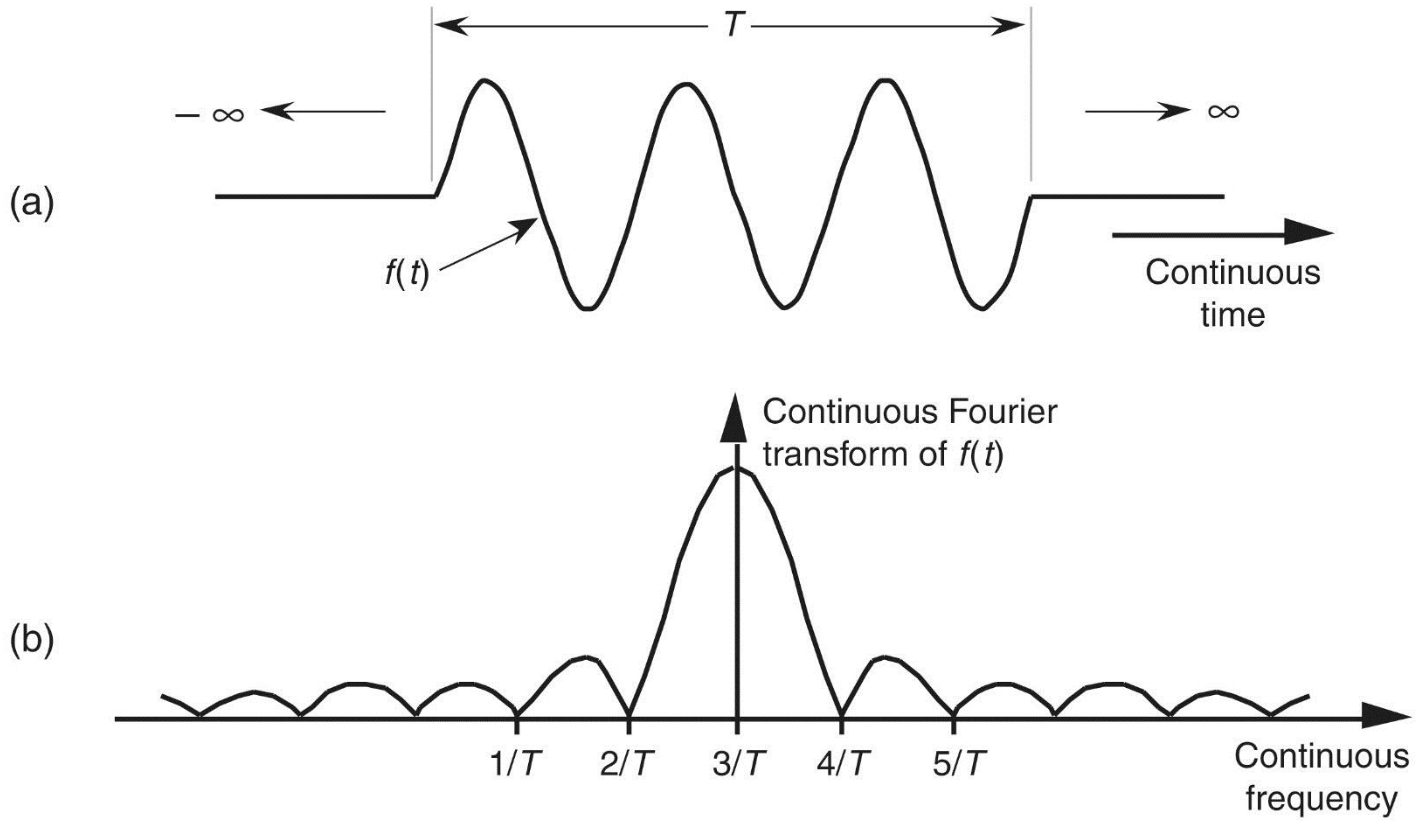


Figure 3–20 Continuous Fourier transform: (a) continuous time-domain $f(t)$ of a truncated sinusoid of frequency $3/T$; (b) continuous Fourier transform of $f(t)$.

DFT Resolution, Zero Padding, Frequency-Domain Sampling

- Fig. 3-20
  - Because CFT is taken over an infinitely wide time interval, CFT has continuous resolution
  - Suppose we want to use a 16-point DFT to approximate CFT of $f(t)$ in Fig. 3-20(a)
    - 16 discrete samples of $f(t)$ are shown on left side of Fig. 3-21(a)
    - Applying those time samples to a 16-point DFT results in discrete frequency-domain samples, the positive frequencies of which are represented on right side of Fig. 3-21(a)
    - DFT output comprises samples of Fig. 3-20(b)'s CFT

DFT Resolution, Zero Padding, Frequency-Domain Sampling

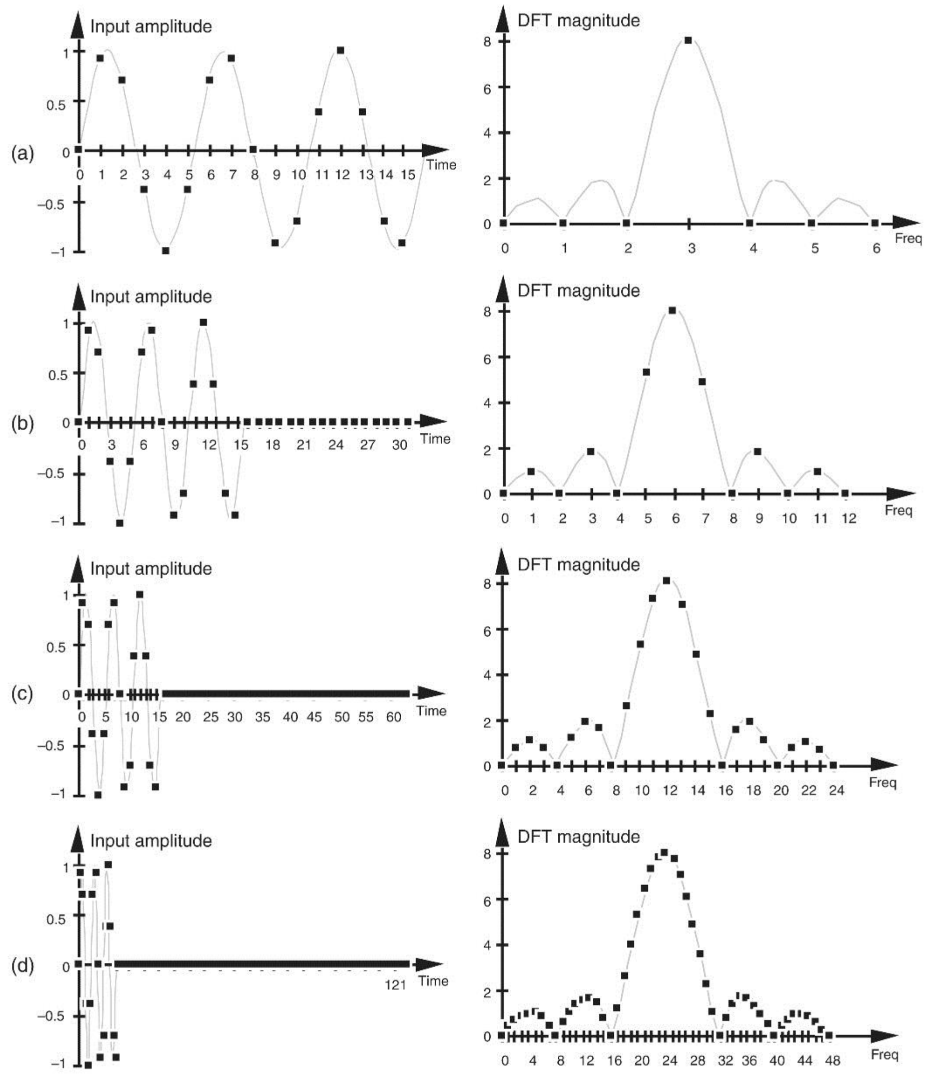


Figure 3–21 DFT frequency-domain sampling: (a) 16 input data samples and $N = 16$; (b) 16 input data samples, 16 padded zeros, and $N = 32$; (c) 16 input data samples, 48 padded zeros, and $N = 64$; (d) 16 input data samples, 112 padded zeros, and $N = 128$.

DFT Resolution, Zero Padding, Frequency-Domain Sampling

- Fig. 3-21
  - If we append 16 zeros to input sequence and take a 32-point DFT, we get output shown on right side of (b)
    - DFT frequency sampling is increased by a factor of two
  - Adding 32 more zeros and taking a 64-point DFT, we get output shown on right side of (c)
    - 64-point DFT output shows true shape of CFT
  - Adding 64 more zeros and taking a 128-point DFT, we get output shown on right side of (d)
    - DFT frequency-domain sampling characteristic is obvious now

DFT Resolution, Zero Padding, Frequency-Domain Sampling

- Fig. 3-21
  - Although zero-padded DFT output bin index of main lobe changes as *N* increases, zero-padded DFT output frequency associated with main lobe remains the same
  - If we perform zero padding on *L* nonzero input samples to get a total of *N* time samples for an *N*-point DFT, zero-padded DFT output bin center frequencies are related to original f_s by

$$\text{center frequency of the } m\text{th bin} = \frac{m f_s}{N}$$

DFT Resolution, Zero Padding, Frequency-Domain Sampling

Fig. no.	**Main lobe peak located at *m* =**	***L* =**	***N* =**	**Frequency of main lobe peak relative to f_s**
3-21(a)	3	16	16	$3f_s / 16$
3-21(b)	6	16	32	$6f_s / 32 = 3f_s / 16$
3-21(c)	12	16	64	$12f_s / 64 = 3f_s / 16$
3-21(d)	24	16	128	$24f_s / 128 = 3f_s / 16$

DFT Resolution, Zero Padding, Frequency-Domain Sampling

- Zero padding
  - DFT magnitude expressions

$$M_{real} = A_o N / 2 \quad \text{and} \quad M_{complex} = A_o N$$

  don't apply if zero padding is used
    - To perform zero padding on *L* nonzero samples of a sinusoid whose frequency is located at a bin center to get a total of *N* input samples, replace *N* with *L* above
  - To perform both zero padding and windowing on input, do not apply window to entire input including appended zero-valued samples
    - Window function must be applied only to original nonzero time samples; otherwise padded zeros will *zero out* and distort part of window function, leading to erroneous results

DFT Resolution, Zero Padding, Frequency-Domain Sampling

- Discrete-time Fourier transform (DTFT)
  - DTFT is continuous Fourier transform of an L-point discrete time-domain sequence
  - On a computer we can't perform DTFT because it has an infinitely fine frequency resolution
    - But we can approximate DTFT by performing an N-point DFT on an L-point discrete time sequence where $N > L$
    - Done by zero-padding original time sequence and taking DFT

DFT Resolution, Zero Padding, Frequency-Domain Sampling

- Zero padding
  - Zero padding does not improve our ability to resolve, to distinguish between, two closely spaced signals in frequency domain
    - E.g., main lobes of various spectra in Fig. 3-21 do *not* change in width, if measured in Hz, with increased zero padding
  - To improve our true spectral resolution of two signals, we need more nonzero time samples
  - To realize F_{res} Hz spectral resolution, we must collect $1/F_{res}$ seconds, worth of nonzero time samples for our DFT processing

DFT Processing Gain

- Two types of processing gain associated with DFTs
  - 1) DFT's *processing gain*
    - Using DFT to detect signal energy embedded in noise
    - DFT can *pull* signals out of background noise
    - This is due to inherent correlation gain that takes place in any N-point DFT
  - 2) *integration gain*
    - Possible when multiple DFT outputs are averaged

DFT Processing Gain

- Processing gain of a single DFT
  - A DFT output bin can be treated as a bandpass filter (band center = mf_s/N) whose gain can be increased and whose bandwidth can be reduced by increasing the value of N
    - Maximum possible DFT output magnitude increases as number of points (N) increases

$$M_{real} = A_o N / 2 \quad \text{and} \quad M_{complex} = A_o N$$

    - Also, as N increases, DFT output bin main lobes become narrower
  - Decreasing a bandpass filter's bandwidth is useful in energy detection because frequency resolution improves in addition to filter's ability to minimize amount of background noise that resides within its passband

DFT Processing Gain

first 32 outputs of a 64-point DFT when input tone is at center of DFT's m = 20th bin

because tone's original signal power is below average noise power level, it is difficult to detect when N = 64

if we quadruple the number of input samples (N = 256), the tone power is raised above average background noise power as shown for m = 80

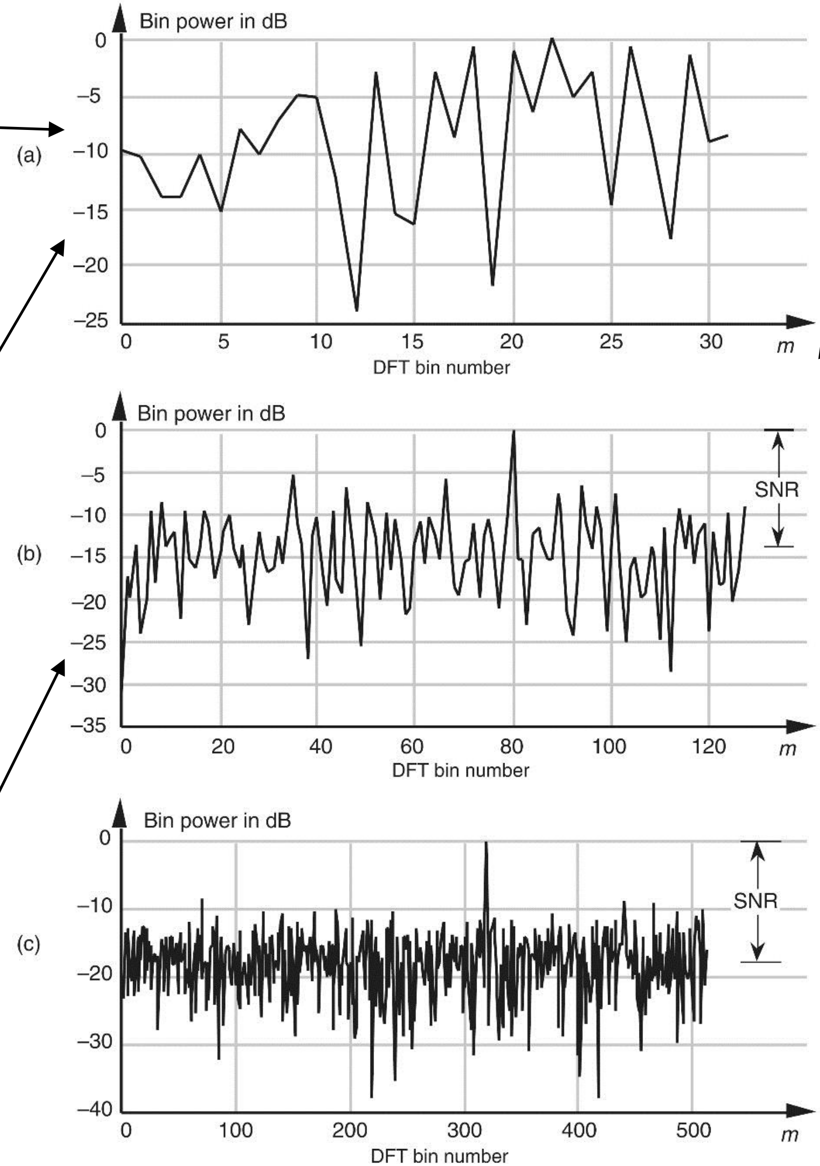


Figure 3-22 Single DFT processing gain: (a) N = 64; (b) N = 256; (c) N = 1024.

DFT of a spectral tone (a constant-frequency sinewave) added to random noise. Output power levels are normalized so that the highest bin output power is set to 0 dB

DFT Processing Gain

- Signal-to-noise ratio (SNR)
  - DFT's *output signal-power level* over the *average output noise-power level*
  - DFT's output SNR increases as N gets larger because a DFT bin's output noise standard deviation (*rms*) value is proportional to $\sqrt{N}$, and DFT's output magnitude for the bin containing signal tone is proportional to N
  - For real inputs, if $N > N'$, an N-point DFT's output SNR_N increases over N'-point DFT $SNR_{N'}$ by:

$$SNR_N = SNR_{N'} + 10\log_{10}(N/N')$$

    - If we increase a DFT's size from N' to $N = 2N'$, DFT's output SNR increases by 3 dB

DFT Processing Gain

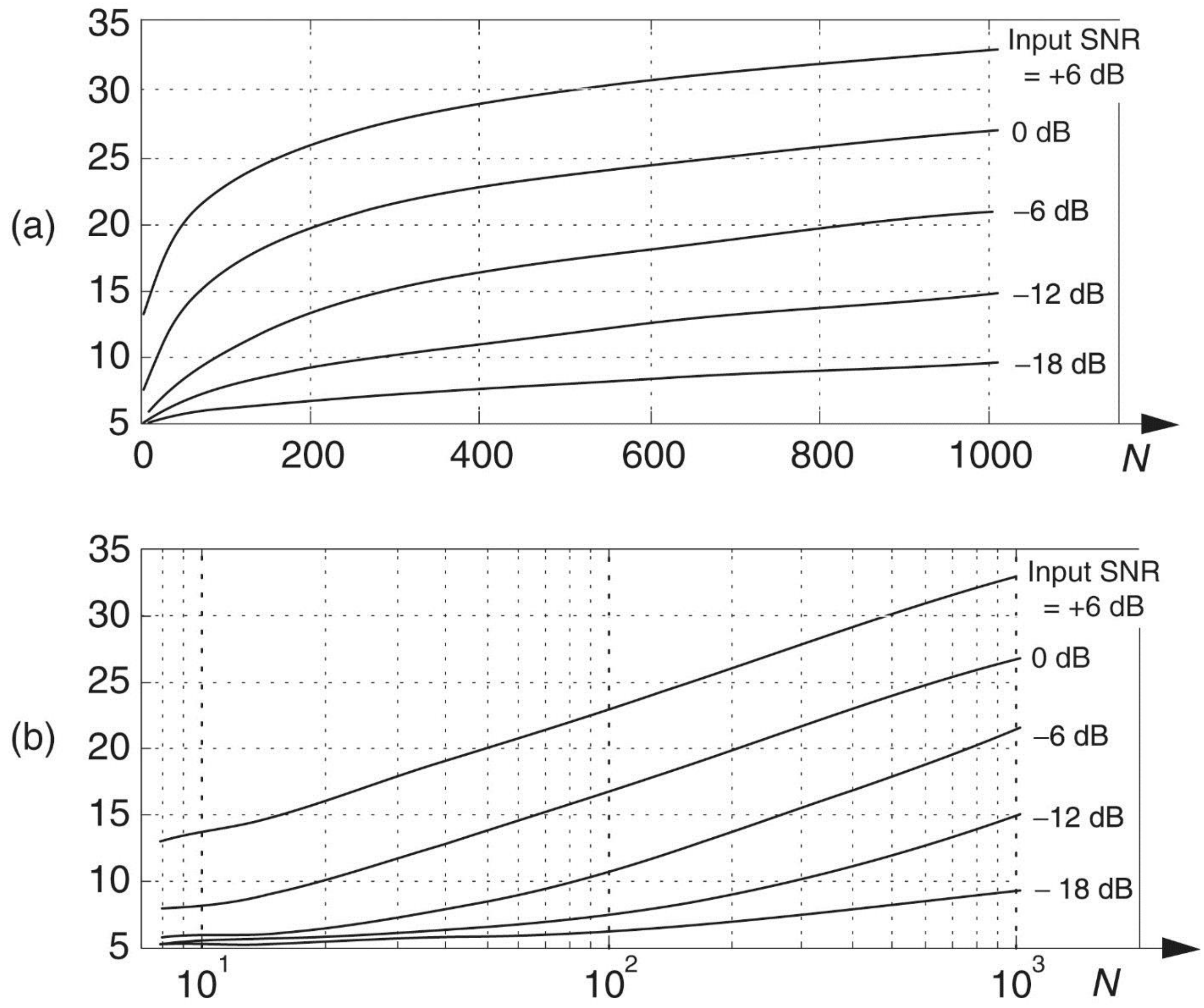


Figure 3–23 DFT processing gain versus number of DFT points *N* for various input signal-to-noise ratios: (a) linear *N* axis; (b) logarithmic *N* axis.

DFT Processing Gain

- Integration gain due to averaging multiple DFTs
  - Theoretically, we could get very large DFT processing gains by increasing DFT size
  - Problem is that the number of necessary DFT multiplications increases proportionally to N^2
    - Larger DFTs become very computationally intensive
  - Because addition is easier and faster to perform than multiplication, we can average outputs of multiple DFTs to obtain further processing gain and signal detection sensitivity

The DFT of Rectangular Functions

- DFT of a rectangular function
  - One of the most prevalent and important computations encountered in DSP
  - Seen in sampling theory, window functions, discussions of convolution, spectral analysis, and in design of digital filters

$$\text{DFT}_{\text{rect. function}} = \frac{\sin(x)}{\sin(x/N)}, \text{ or } \frac{\sin(x)}{x}, \text{ or } \frac{\sin(Nx/2)}{\sin(x/2)}$$

The DFT of Rectangular Functions

- DFT of a general rectangular function
  - A general rectangular function $x(n)$ is defined as N samples containing K unity-valued samples

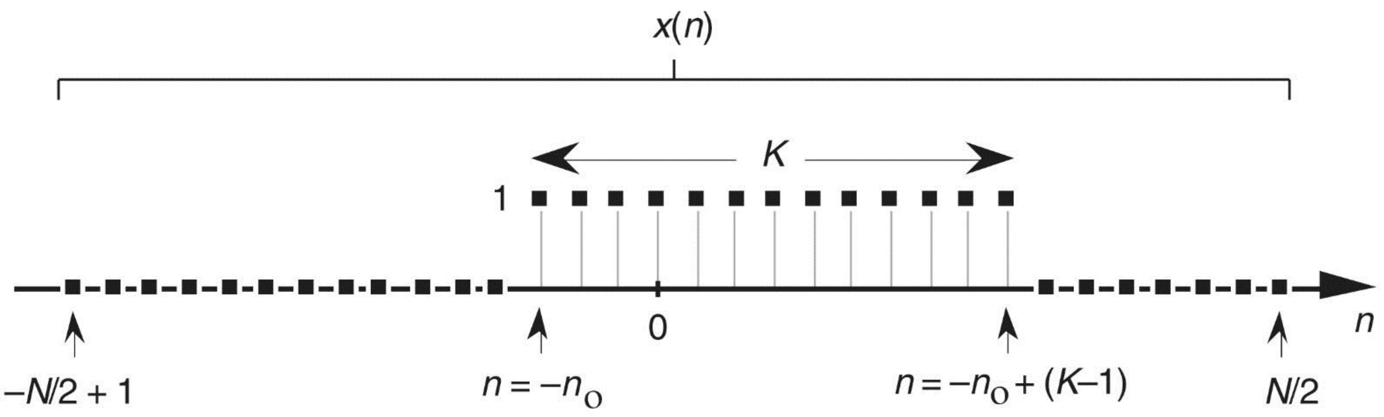


Figure 3-24 Rectangular function of width K samples defined over N samples where $K < N$.

The DFT of Rectangular Functions

$$X(m) = \sum_{n=-(N/2)+1}^{N/2} x(n)\, e^{-j2\pi nm/N}$$

$$= \sum_{n=-n_o}^{-n_o+(K-1)} 1 \cdot e^{-j2\pi nm/N}$$

$$\xrightarrow{q=2\pi m/N}$$

$$X(q) = \sum_{n=-n_o}^{-n_o+(K-1)} e^{-jqn}$$

$$= e^{-jq(-n_o)} + e^{-jq(-n_o+1)} + e^{-jq(-n_o+2)} + \ldots + e^{-jq(-n_o+(K-1))}$$

$$= e^{-jq(-n_o)} e^{-j0q} + e^{-jq(-n_o)} e^{-j1q} + e^{-jq(-n_o)} e^{-j2q} + \ldots + e^{-jq(-n_o)} e^{-jq(K-1)}$$

$$= e^{jq(n_o)} \cdot [e^{-j0q} + e^{-j1q} + e^{-j2q} + \ldots + e^{-jq(K-1)}]$$

$$X(q) = e^{jq(n_o)} \sum_{p=0}^{K-1} e^{-jpq}$$

The DFT of Rectangular Functions

$$X(q) = e^{jq(n_o)} \underbrace{\sum_{p=0}^{K-1} e^{-jpq}}_{\text{geometric series}}$$

$$\sum_{p=0}^{K-1} e^{-jpq} = \frac{1-e^{-jqK}}{1-e^{-jq}}$$

$$= \frac{e^{-jqK/2}(e^{jqK/2} - e^{-jqK/2})}{e^{-jq/2}(e^{jq/2} - e^{-jq/2})}$$

$$= e^{-jq(K-1)/2} \cdot \frac{(e^{jqK/2} - e^{-jqK/2})}{(e^{jq/2} - e^{-jq/2})}$$

$$\xrightarrow[\sin(\phi)=(e^{j\phi}-e^{-j\phi})/2j]{\text{Euler's equation:}} = e^{-jq(K-1)/2} \cdot \frac{2j\sin(qK/2)}{2j\sin(q/2)}$$

$$\sum_{p=0}^{K-1} e^{-jpq} = e^{-jq(K-1)/2} \cdot \frac{\sin(qK/2)}{\sin(q/2)}$$

The DFT of Rectangular Functions

$$X(q) = e^{jq(n_o)} \sum_{p=0}^{K-1} e^{-jpq}$$

$$\xrightarrow{\sum_{p=0}^{K-1} e^{-jpq} = e^{-jq(K-1)/2} \cdot \frac{\sin(qK/2)}{\sin(q/2)}} = e^{jq(n_o)} \cdot e^{-jq(K-1)/2} \cdot \frac{\sin(qK/2)}{\sin(q/2)}$$

$$= e^{jq(n_o-(K-1)/2)} \cdot \frac{\sin(qK/2)}{\sin(q/2)}$$

$$\xrightarrow{q=2\pi m/N} X(m) = e^{j(2\pi m/N)(n_o-(K-1)/2)} \cdot \frac{\sin(2\pi mK/2N)}{\sin(2\pi m/2N)}$$

$$\xrightarrow{\text{General form of the Dirichlet kernel}} X(m) = e^{j(2\pi m/N)(n_o-(K-1)/2)} \cdot \frac{\sin(\pi mK/N)}{\sin(\pi m/N)}$$

The DFT of Rectangular Functions

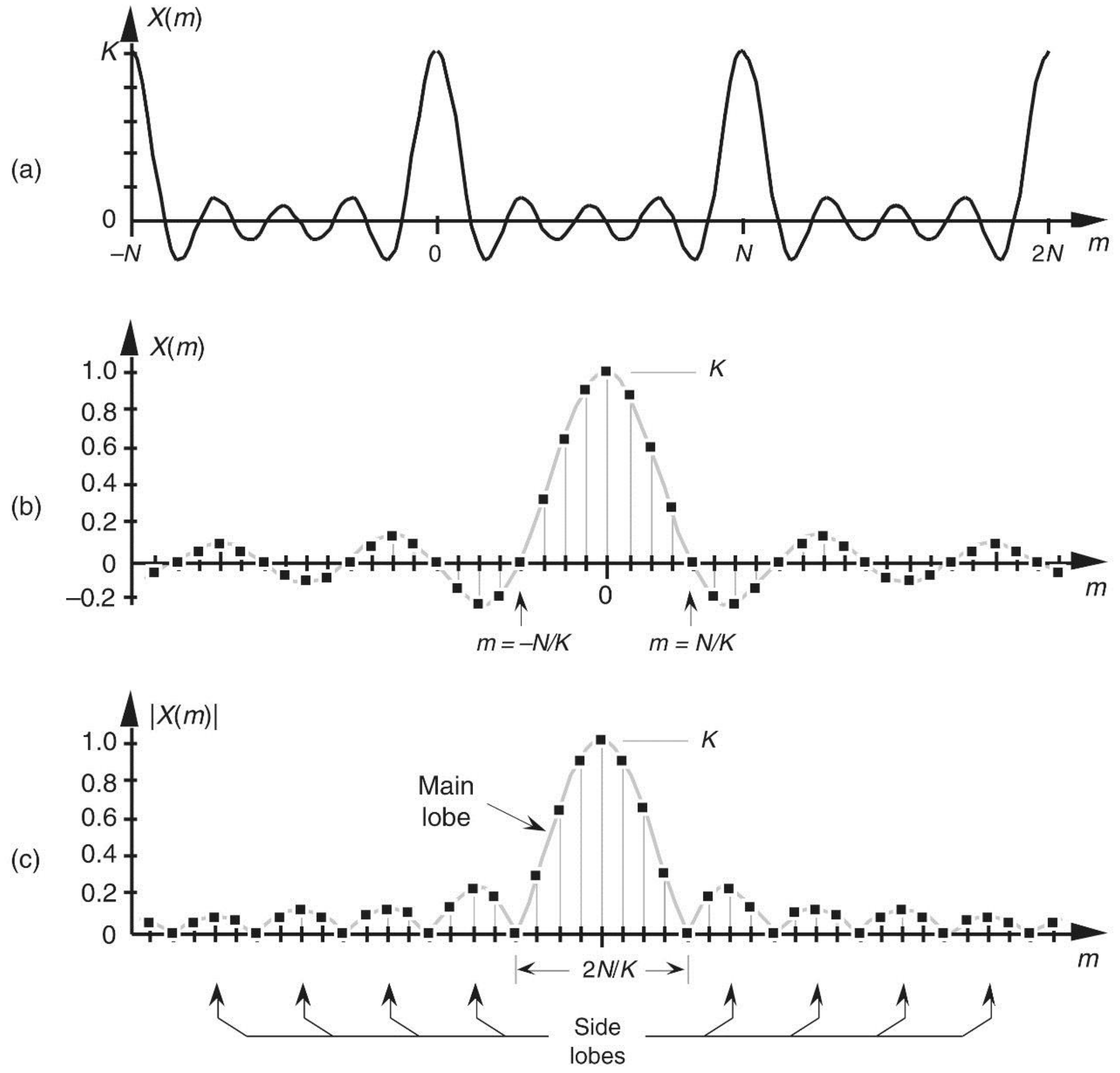


Figure 3–25 The Dirichlet kernel of $X(m)$: (a) periodic continuous curve on which the $X(m)$ samples lie; (b) $X(m)$ amplitudes about the $m = 0$ sample; (c) $|X(m)|$ magnitudes about the $m = 0$ sample.

The DFT of Rectangular Functions

- Dirichlet kernel (DFT of rectangular function)
  - Has a main lobe, centered about *m* = 0 point
  - Peak amplitude of main lobe is *K*
    - *X*(0) = sum of *K* unity-valued samples = *K*
  - Main lobe's width = 2*N*/*K*

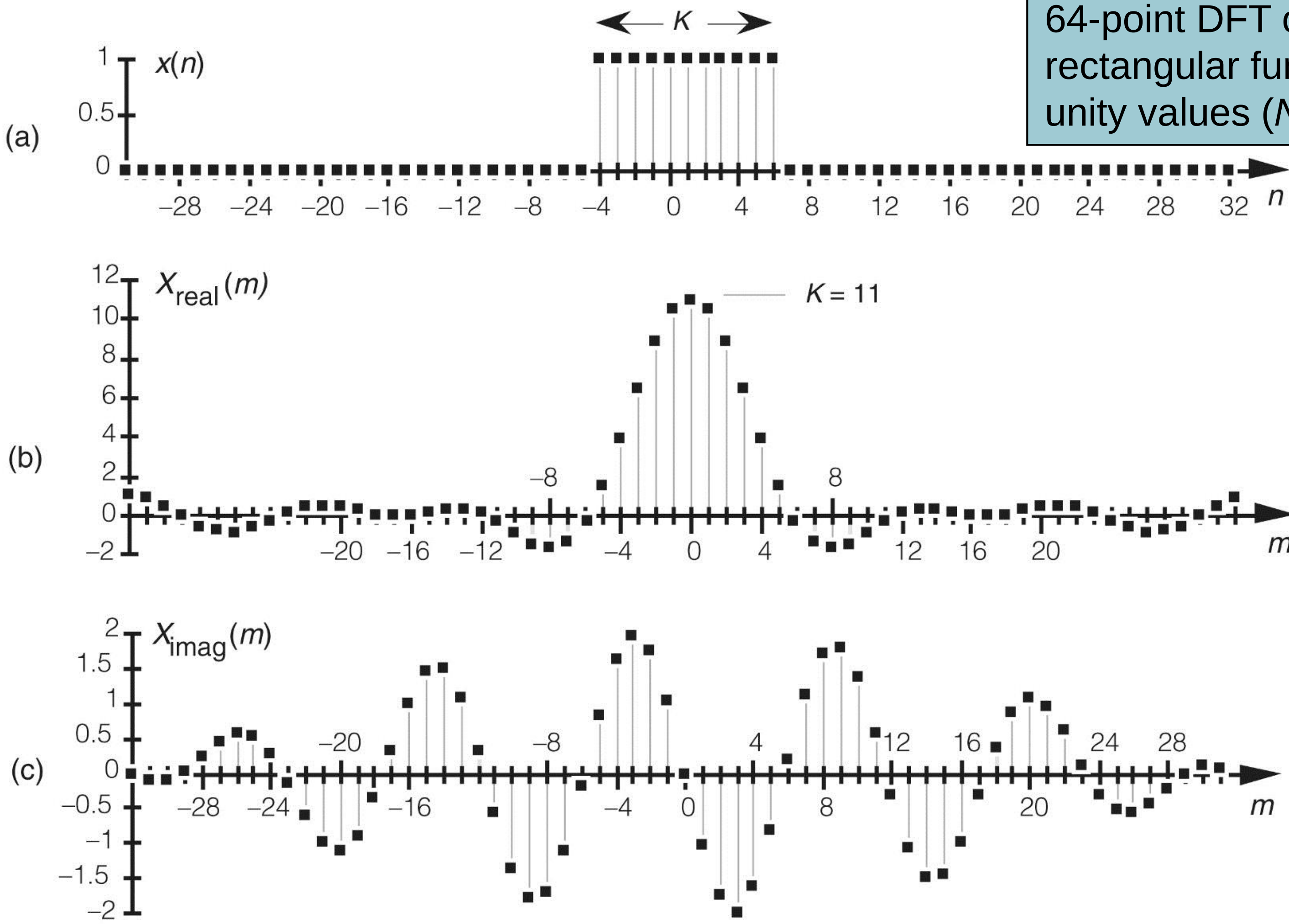
$$X(m) = e^{j(2\pi m/N)(n_o-(K-1)/2)} \cdot \frac{\sin(\overbrace{\pi mK/N}^{=\pi})}{\sin(\pi m/N)}$$

$$m_{\text{first zero crossing}} = \frac{\pi N}{\pi K} = \frac{N}{K}$$

    - Thus main lobe width is inversely proportional to *K*
    - A fundamental characteristic of Fourier transforms: the narrower the function in one domain, the wider its transform will be in the other domain

The DFT of Rectangular Functions

64-point DFT of 64-sample rectangular function, with 11 unity values (N = 64 and K = 11)



it's easier to comprehend the true spectral nature of $X(m)$ by viewing its absolute magnitude, provided in Fig. 3-27(a)

Figure 3–26 DFT of a rectangular function: (a) original function $x(n)$; (b) real part of the DFT of $x(n)$, $X_{real}(m)$; (c) imaginary part of the DFT of $x(n)$, $X_{imag}(m)$.

The DFT of Rectangular Functions

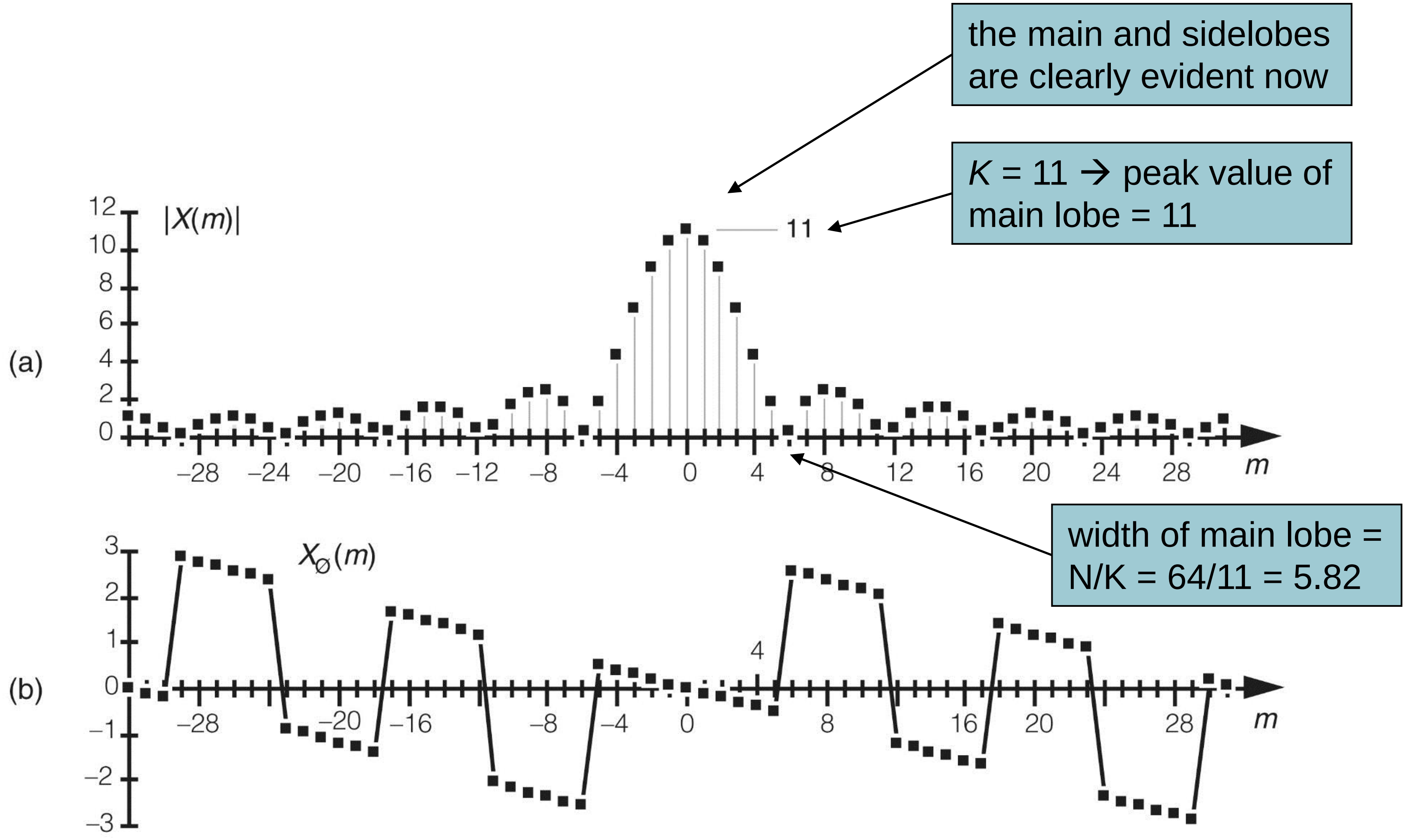


Figure 3–27 DFT of a generalized rectangular function: (a) magnitude $|X(m)|$; (b) phase angle in radians.

The DFT of Rectangular Functions

- DFT of a symmetrical rectangular function

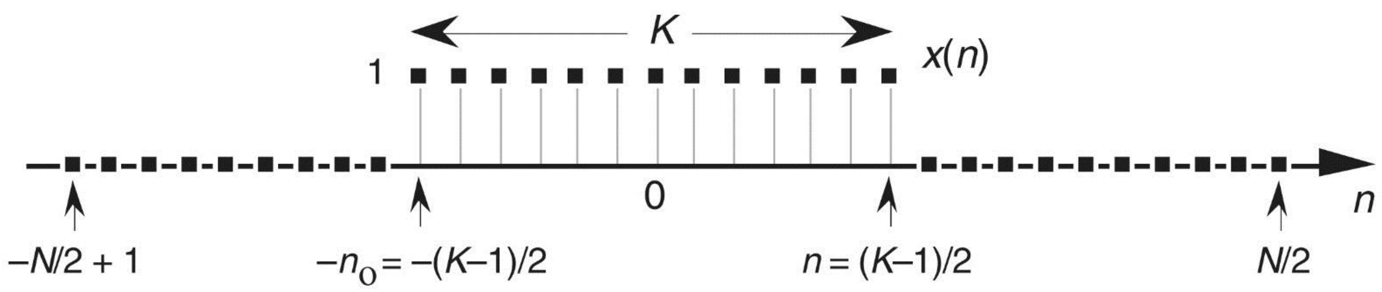


Figure 3–28 Rectangular $x(n)$ with K samples centered about $n = 0$.

$$X(m) = e^{j(2\pi m/N)(n_o-(K-1)/2)} \cdot \frac{\sin(\pi mK/N)}{\sin(\pi m/N)}$$

$$\xrightarrow{n_o=(K-1)/2} X(m) = e^{j(2\pi m/N)((K-1)/2-(K-1)/2)} \cdot \frac{\sin(\pi mK/N)}{\sin(\pi m/N)}$$

$$= e^{j(2\pi m/N)(0)} \cdot \frac{\sin(\pi mK/N)}{\sin(\pi m/N)}$$

$$\xrightarrow{\text{Symmetrical form of the Dirichlet kernel}} X(m) = \frac{\sin(\pi mK/N)}{\sin(\pi m/N)}$$

The DFT of Rectangular Functions

- DFT of a symmetrical rectangular function

$$X(m) = \frac{\sin(\pi m K / N)}{\sin(\pi m / N)}$$

  - This DFT is itself a real function
    - So it contains no imaginary part or phase term
    - If $x(n)$ is real and even, $x(n) = x(-n)$, then $X_{real}(m)$ is nonzero and $X_{imag}(m)$ is always zero

The DFT of Rectangular Functions

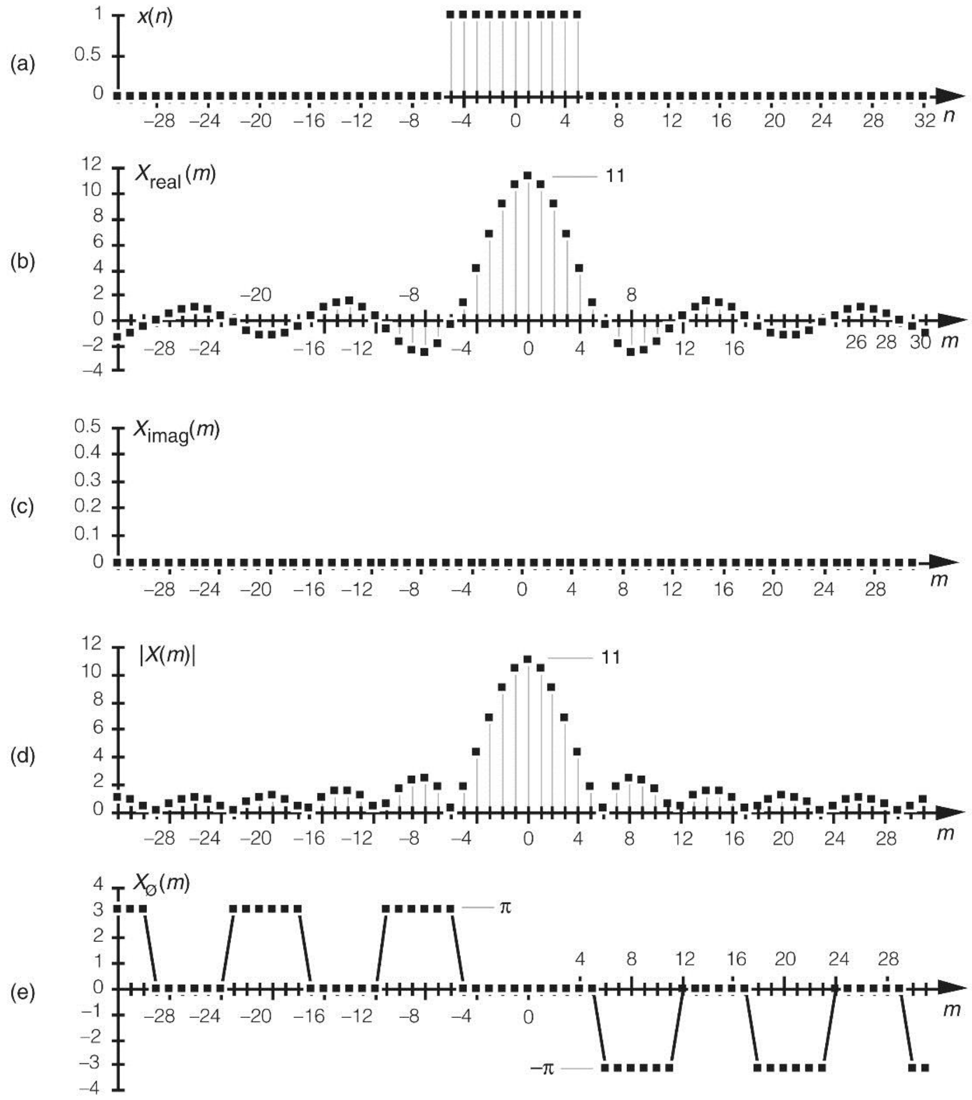


Figure 3–29 DFT of a rectangular function centered about $n = 0$: (a) original $x(n)$; (b) $X_{real}(m)$; (c) $X_{imag}(m)$; (d) magnitude of $X(m)$; (e) phase angle of $X(m)$ in radians.

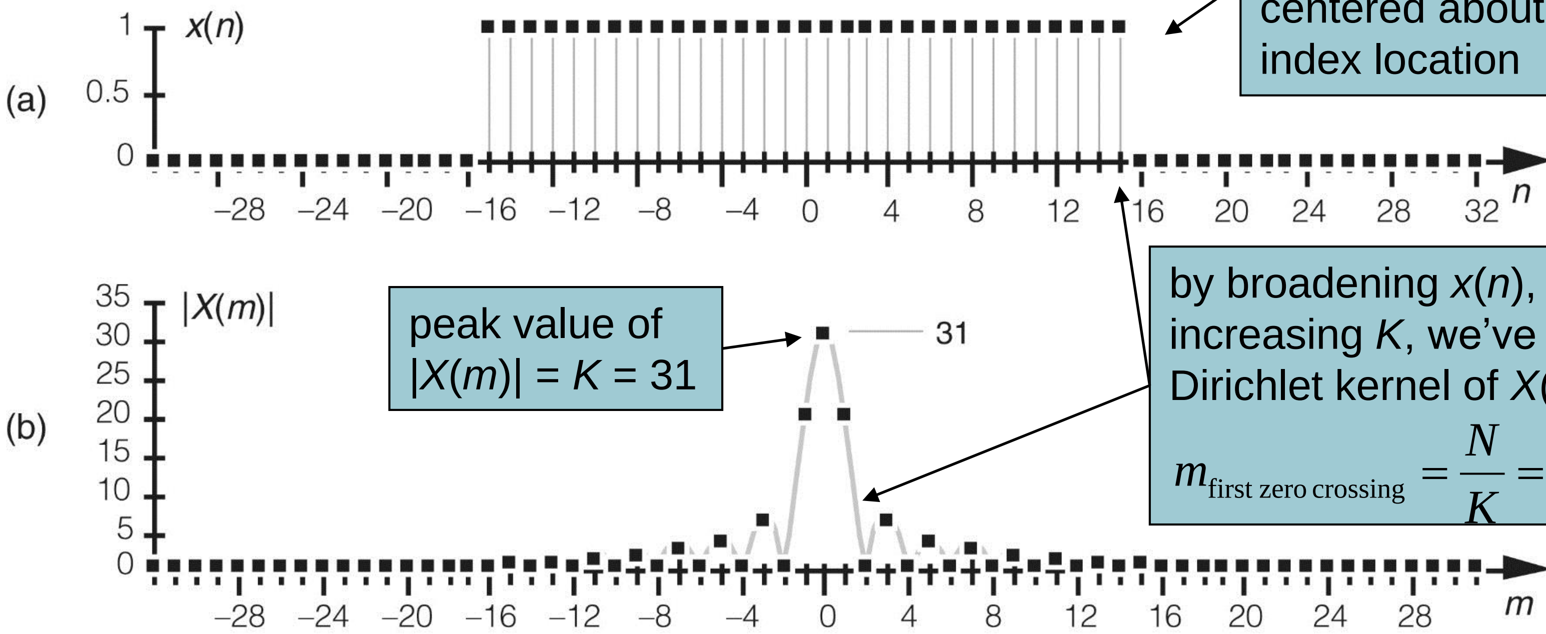
The DFT of Rectangular Functions

- Fig. 3-29 (64-point DFT)
  - $X_{real}(m)$ is nonzero and $X_{imag}(m)$ is zero
  - Identical magnitudes in Figs. 3-27(a) and 3-29(d)
    - Shifting K unity-valued samples to center merely affects phase angle of $X(m)$, not its magnitude (shifting theorem of DFT)
  - Even though $X_{imag}(m)$ is zero in (c), phase angle of $X(m)$ is not always zero
    - $X(m)$'s phase angles in (e) are either $+\pi$, zero, or $-\pi$
    - $e^{j\pi} = e^{j(-\pi)} = -1$ → we could easily reconstruct $X_{real}(m)$ from $|X(m)|$ and phase angle $X_{ø}(m)$ if we must
    - $X_{real}(m)$ is equal to $|X(m)|$ with the signs of $|X(m)|$'s alternate sidelobes reversed

The DFT of Rectangular Functions

another example of how DFT of a rectangular function is a sampled version of Dirichlet kernel

a 64-point $x(n)$ where 31 unity-valued samples are centered about $n = 0$ index location



peak value of $|X(m)| = K = 31$

by broadening $x(n)$, i.e., increasing K, we've narrowed Dirichlet kernel of $X(m)$

$$m_{\text{first zero crossing}} = \frac{N}{K} = \frac{64}{31}$$

Figure 3–30 DFT of a symmetrical rectangular function with 31 unity values: (a) original $x(n)$; (b) magnitude of $X(m)$.

The DFT of Rectangular Functions

- DFT of an all-ones rectangular function

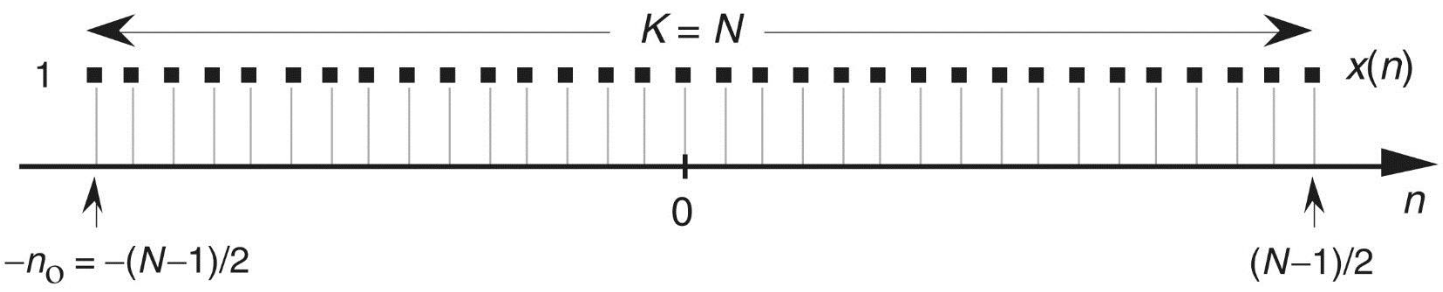


Figure 3-31 Rectangular function with *N* unity-valued samples.

$$X(m) = e^{j(2\pi m/N)(n_o-(K-1)/2)} \cdot \frac{\sin(\pi mK/N)}{\sin(\pi m/N)}$$

$$\xrightarrow[n_o=(N-1)/2]{K=N \text{ and}} X(m) = e^{j(2\pi m/N)((N-1)/2-(N-1)/2)} \cdot \frac{\sin(\pi mN/N)}{\sin(\pi m/N)}$$

$$= e^{j(2\pi m/N)(0)} \cdot \frac{\sin(\pi m)}{\sin(\pi m/N)}$$

$$\xrightarrow[\text{Dirichlet kernel (Type 1)}]{\text{All-ones form of the}} X(m) = \frac{\sin(\pi m)}{\sin(\pi m/N)}$$

The DFT of Rectangular Functions

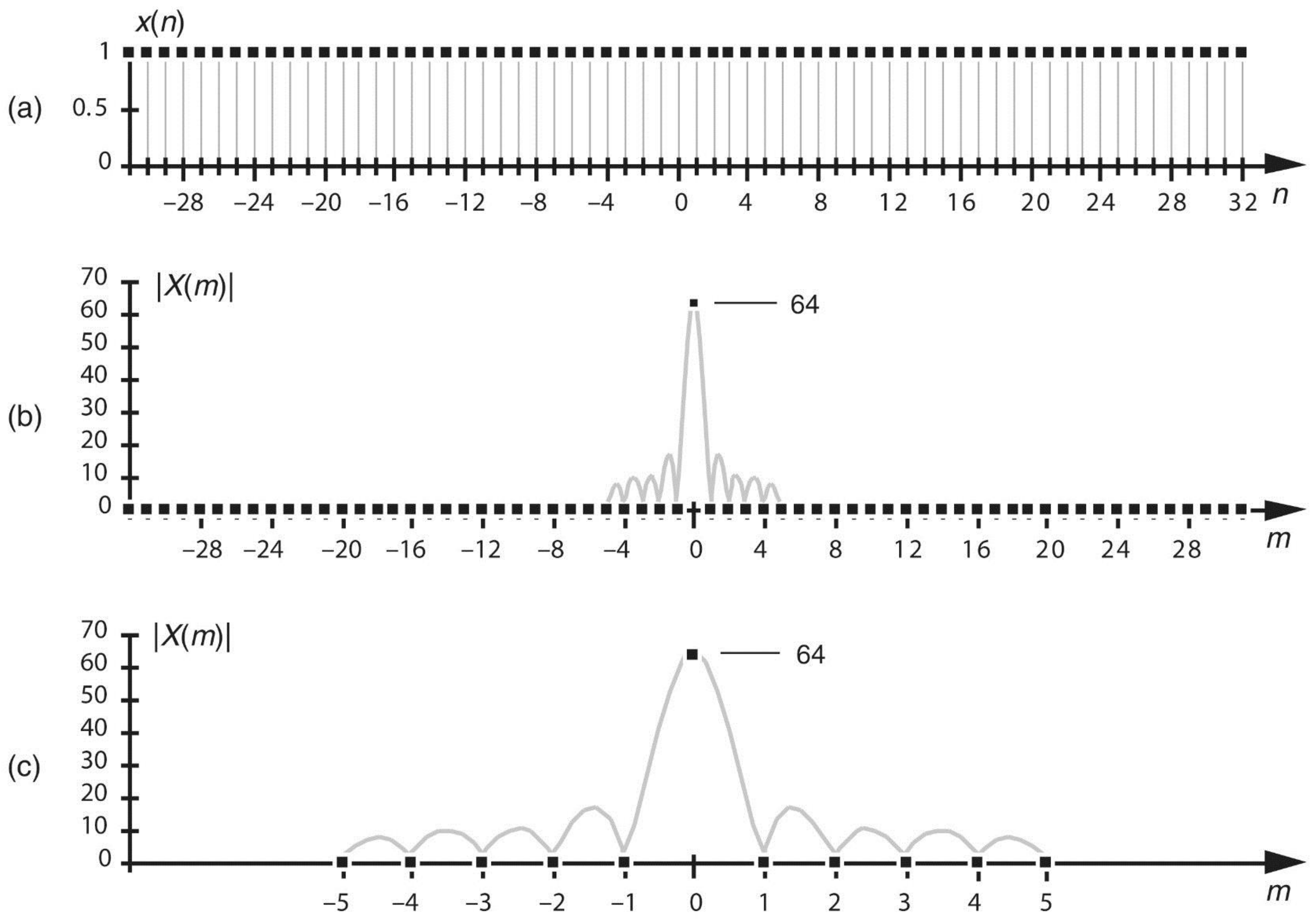


Figure 3–32 All-ones function: (a) rectangular function with $N = 64$ unity-valued samples; (b) DFT magnitude of the all-ones time function; (c) close-up view of the DFT magnitude of an all-ones time function.

The DFT of Rectangular Functions

- Fig. 3-32
  - Dirichlet kernel of $X(m)$ in (b) is as narrow as it can get
  - Main lobe's first positive zero crossing occurs at $m = 64/64 = 1$ sample in (b)
  - Peak value of $|X(m)| = N = 64$
  - $x(n)$ is all ones → $|X(m)|$ is zero for all $m \neq 0$
  - The sinc function

$$\xrightarrow[\text{Dirichlet kernel (Type1)}]{\text{All-ones form of the}} X(m) = \frac{\sin(\pi m)}{\sin(\pi m / N)}$$

    - Defines overall DFT frequency response to an input sinusoidal sequence
    - Is also amplitude response of a single DFT bin

The DFT of Rectangular Functions

- DFT of an all-ones rectangular function

$$\xrightarrow[\text{}]{\substack{\text{All-ones form of the} \\ \text{Dirichlet kernel (Type 1)}}} X(m) = \frac{\sin(\pi m)}{\underbrace{\sin(\pi m / N)}_{\alpha \text{ is small} \rightarrow \sin(\alpha) \approx \alpha}}$$

$$\xrightarrow[\text{}]{\substack{\text{All-ones form of the} \\ \text{Dirichlet kernel (Type 2)} \\ \text{(when N is large)}}} X(m) \approx \frac{\sin(\pi m)}{\pi m / N} = N \cdot \frac{\sin(\pi m)}{\pi m}$$

$$\xrightarrow[\text{}]{\substack{\text{All-ones form of the} \\ \text{Dirichlet kernel (Type 3)} \\ \text{(normalized)}}} X(m) \approx \frac{\sin(\pi m)}{\pi m}$$

The DFT of Rectangular Functions

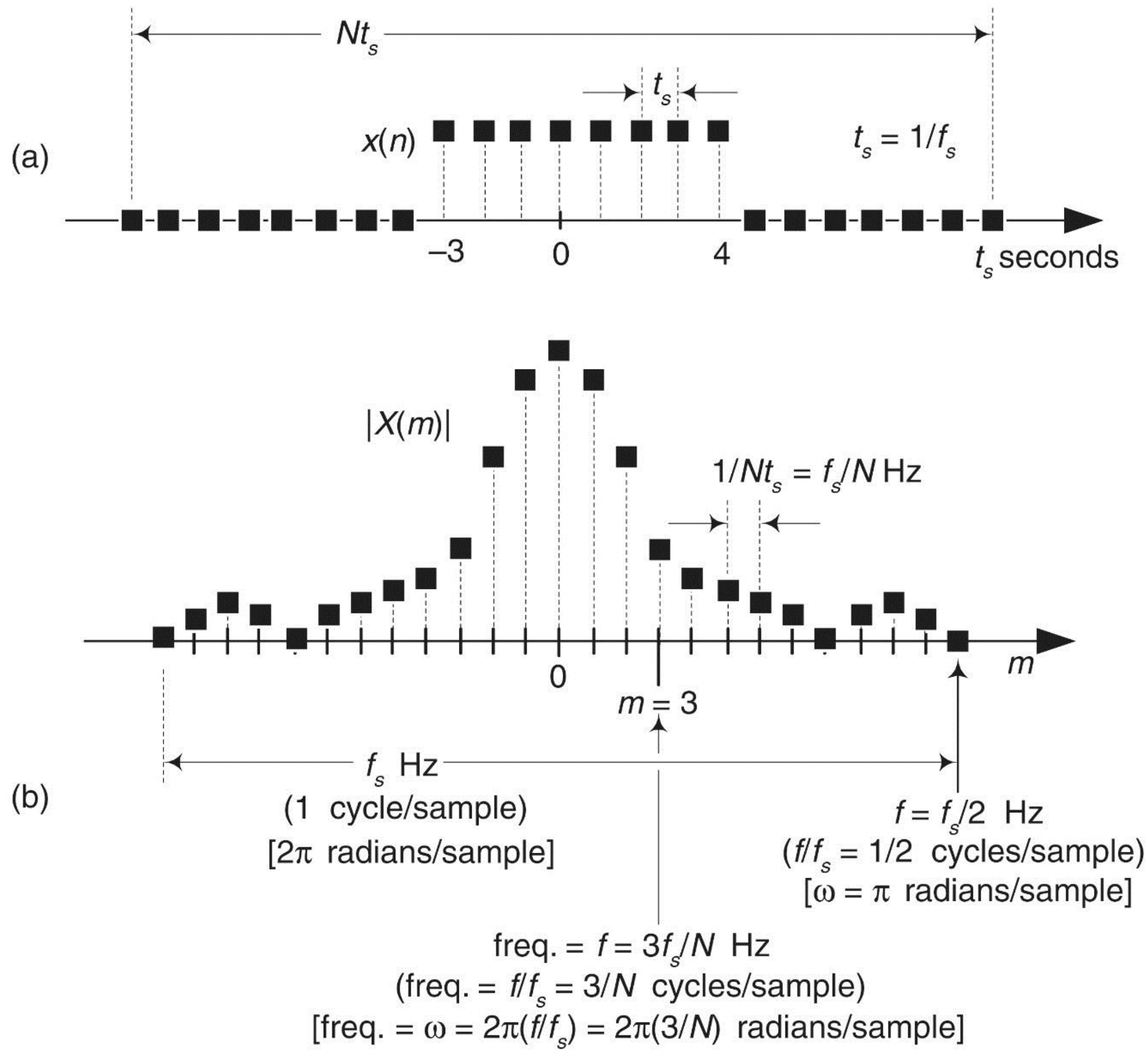


Figure 3–34 DFT time and frequency axis dimensions: (a) time-domain axis uses time index *n*; (b) various representations of the DFT's frequency axis.

The DFT of Rectangular Functions

DFT frequency axis representation	Frequency variable	Resolution of X(m)	Repetition interval of X(m)	Frequency axis range
Frequency in Hz	f	f_s/N	f_s	$-f_s/2$ to $f_s/2$
Frequency in cycles/sample	f/f_s	$1/N$	1	$-1/2$ to $1/2$
Frequency in radians/sample	ω	$2\pi/N$	2π	$-\pi$ to π

The DFT of Rectangular Functions

- Alternate form of DFT of an all-ones rectangular function
  - Using radians/sample frequency notation for DFT axis leads to another prevalent form of DFT of all-ones rectangular function
  - Letting normalized discrete frequency axis variable be ω = 2π*m*/*N*, then π*m* = *N*ω/2

$$\xrightarrow{\text{All-ones form of the Dirichlet kernel (Type 1)}} X(m) = \frac{\sin(\pi m)}{\sin(\pi m / N)}$$

$$\xrightarrow{\text{All-ones form of the Dirichlet kernel (Type 4)}} X(\omega) = \frac{\sin(N\omega / 2)}{\sin(\omega / 2)}$$

Interpreting DFT Using DTFT

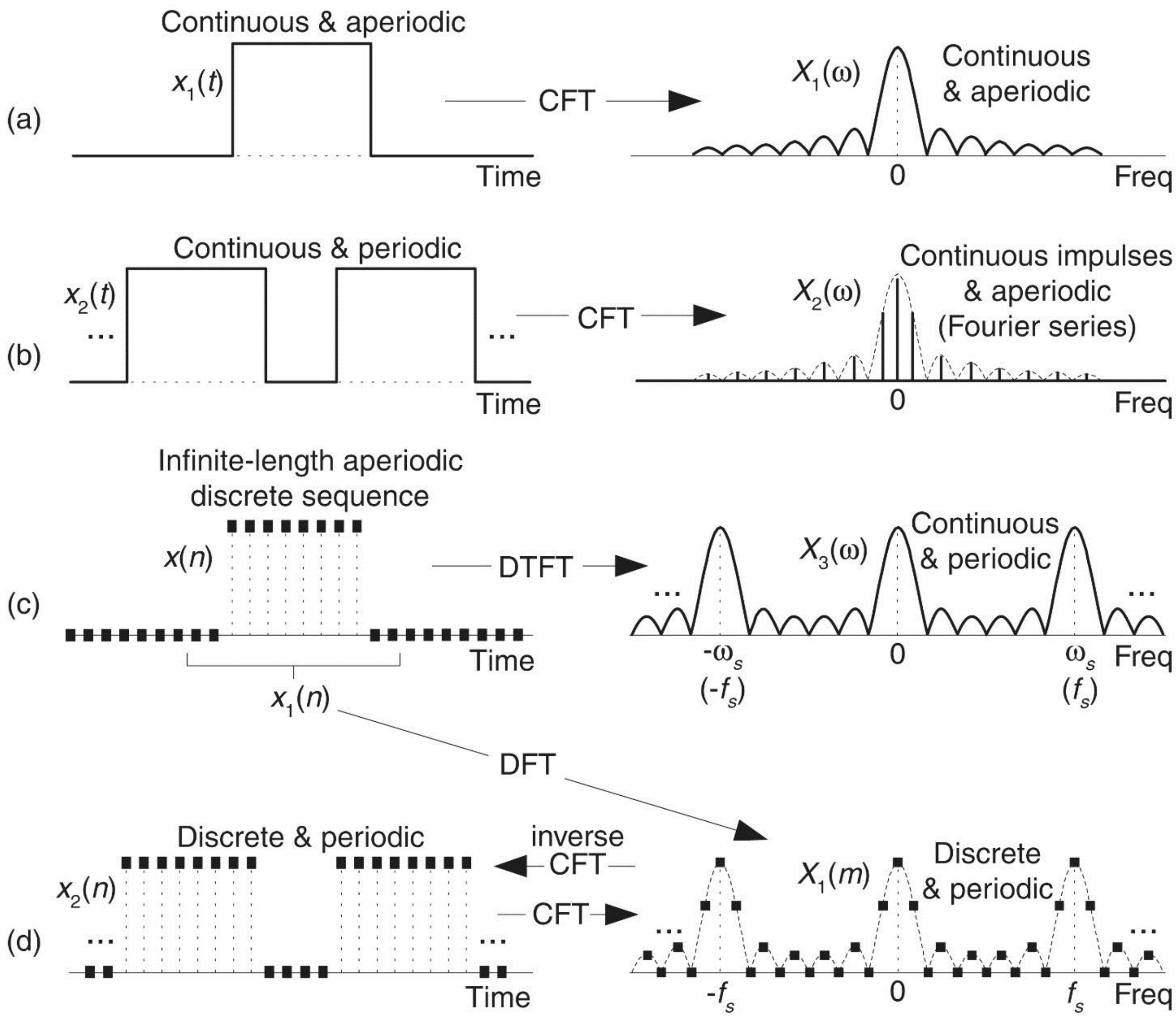


Figure 3–35 Time-domain signals and sequences, and the magnitudes of their transforms in the frequency domain.

Interpreting DFT Using DTFT

- Fig. 3-35
  - (a) shows an infinite-length continuous-time signal containing a single finite-width pulse
    - Magnitude of its continuous Fourier transform (CFT) is continuous frequency-domain function $X_1(\omega)$
    - continuous frequency variable ω is radians per second
  - If CFT is performed on infinite-length signal of periodic pulses in (b), result is line spectra known as *Fourier series* $X_2(\omega)$
    - $X_2(\omega)$ Fourier series is a sampled version of continuous spectrum in (a)

Interpreting DFT Using DTFT

- Fig. 3-35
  - (c) shows infinite-length discrete time sequence $x(n)$, containing several nonzero samples
    - We can perform a CFT of $x(n)$ describing its spectrum as a continuous frequency-domain function $X_3(\omega)$
    - This continuous spectrum is called a discrete-time Fourier transform (DTFT) defined by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

    - ω frequency variable is measured in radians/sample

Interpreting DFT Using DTFT

- DTFT example
  - Time sequence: $x_o(n) = (0.75)^n$ for $n \geq 0$
  - Its DTFT is

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$X_o(\omega) = \sum_{n=0}^{\infty} 0.75^n e^{-j\omega n} = \sum_{n=0}^{\infty} (0.75 e^{-j\omega})^n$$

$$\xrightarrow{\text{geometric series}} X_o(\omega) = \frac{1}{1-0.75e^{-j\omega}} = \frac{e^{j\omega}}{e^{j\omega}-0.75}$$

  - $X_o(\omega)$ is continuous and periodic with a period of 2π, whose magnitude is shown in Fig. 3-36

Interpreting DFT Using DTFT

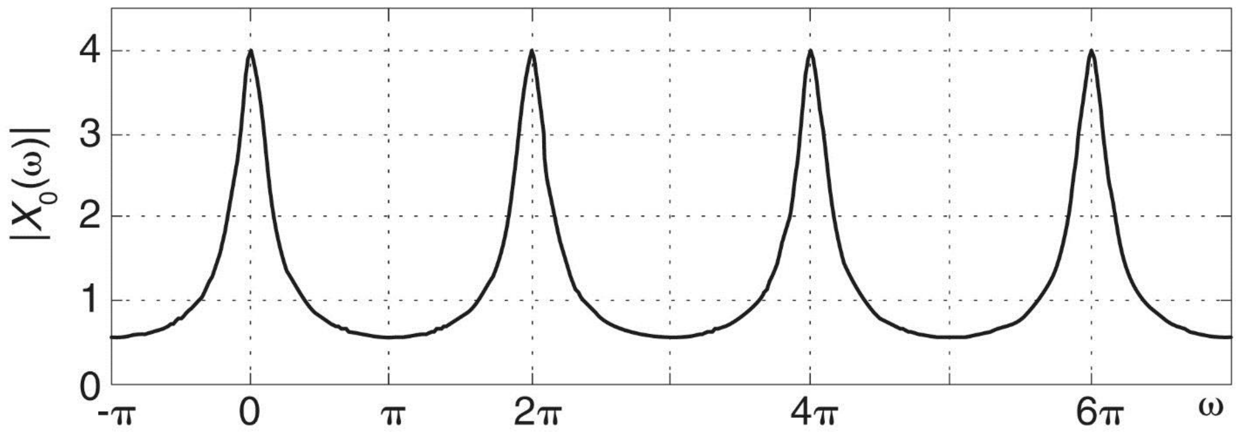


Figure 3-36 DTFT magnitude $|X_O(\omega)|$.

- Verification of 2π periodicity of DTFT

$$X(\omega+2\pi k)=\sum_{n=-\infty}^{\infty}x(n)e^{-j(\omega+2\pi k)n}=\sum_{n=-\infty}^{\infty}x(n)e^{-j\omega n}\overbrace{e^{-j2\pi kn}}^{=1}$$

$$=\sum_{n=-\infty}^{\infty}x(n)e^{-j\omega n}=X(\omega)$$

Interpreting DFT Using DTFT

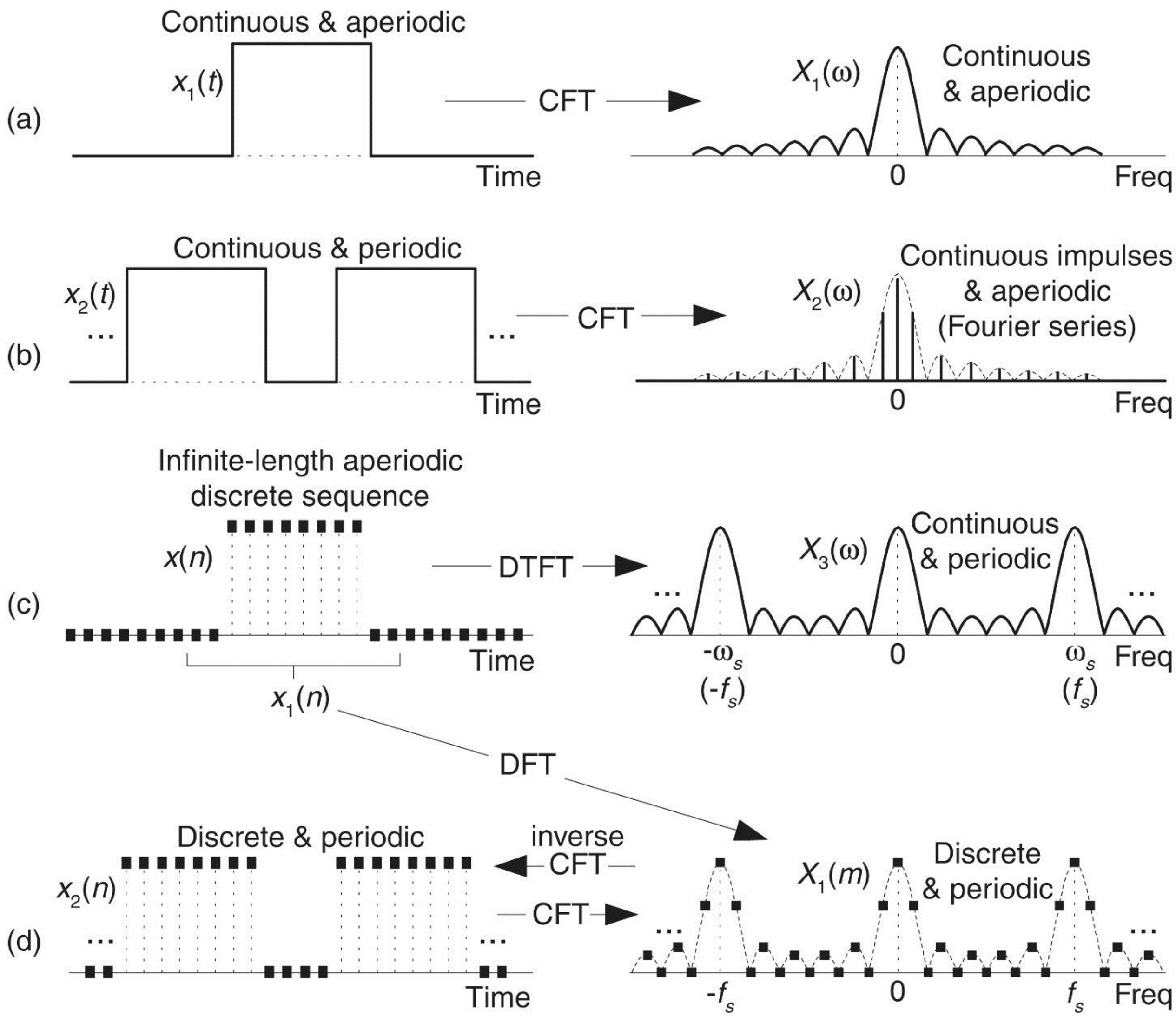


Figure 3–35 Time-domain signals and sequences, and the magnitudes of their transforms in the frequency domain.

Interpreting DFT Using DTFT

- Fig. 3-35 (cont.)
  - Transforms indicated in Figs. (a) through (c) are pencil-and-paper mathematics of calculus
  - In a computer, using only finite-length discrete sequences, we can only approximate CFT (the DTFT) of infinite-length $x(n)$ time sequence in (c)
    - That approximation is DFT, and it's the only Fourier transform tool available
    - Taking DFT of $x_1(n)$, where $x_1(n)$ is a finite-length portion of $x(n)$, we obtain discrete periodic $X_1(m)$ in (d)
    - $X_1(m)$ is a sampled version of $X_3(\omega)$

$$X_1(m) = X_3(\omega)\big|_{\omega=2\pi m/N} = \sum_{n=0}^{N-1} x_1(n)\, e^{-j2\pi nm/N}$$

Interpreting DFT Using DTFT

- Fig. 3-35
  - $X_1(m)$ is *also* exactly equal to CFT of periodic time sequence $x_2(n)$ in (d)
  - *The DFT is equal to the continuous Fourier transform (the DTFT) of a periodic time-domain discrete sequence*
  - If a function is periodic, its forward/inverse DTFT will be discrete; if a function is discrete, its forward/inverse DTFT will be periodic