Digital Signal Processing

The Arithmetic of Complex Numbers

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Graphical Representation of Real and Complex Numbers

- Real number
  - Can be represented by a point on a one-dimensional axis, called *real* axis

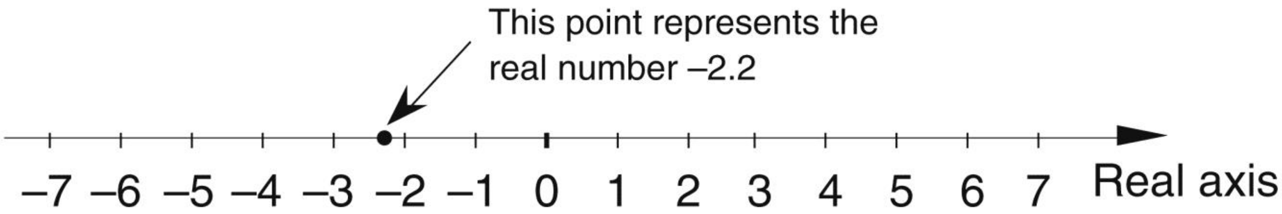

Figure A-1 The representation of a real number as a point on the one-dimensional real axis.

Graphical Representation of Real and Complex Numbers

- Complex number
  - Has two parts: a real part and an imaginary part
  - Can be treated as a point on a complex plane

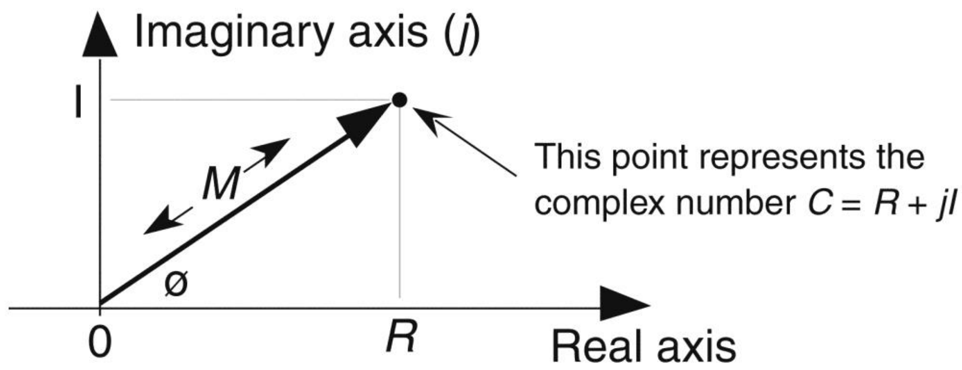

Figure A-2 The phasor representation of the complex number $C = R + jI$ on the complex plane.

Arithmetic Representation of Complex Numbers

- A complex number *C* is represented in a number of different ways
  - Rectangular form

$$C = R + jI$$

  - Trigonometric form

$$C = M[\cos(\phi) + j\sin(\phi)]$$

  - Exponential form

$$C = Me^{j\phi}$$

  - Magnitude and angle form

$$C = M\angle\phi$$

Arithmetic Representation of Complex Numbers

- Magnitude (*modulus*) of *C*

$$M = |C| = \sqrt{R^2 + I^2}$$

- Phase angle (*argument*) of *C*

$$\phi = \tan^{-1}\left(\frac{I}{R}\right)$$

- In exponential form

$$C = Me^{j\phi} = Me^{j(\phi + 2\pi n)}$$

- Variable ø need not be constant

$$C = Me^{j\omega t} \quad \text{or} \quad C = Me^{-j\omega t}$$

  - A phasor of magnitude *M* that rotates in a (counter)clockwise direction at a radian frequency of (+ω) –ω radians per second

Arithmetic Operations of Complex Numbers

- Addition and subtraction
  - Rectangular form is the easiest to use here

$$C_1 + C_2 = R_1 + jI_1 + R_2 + jI_2 = R_1 + R_2 + j(I_1 + I_2)$$

$$C_1 - C_2 = (R_1 + jI_1) - (R_2 + jI_2) = R_1 - R_2 + j(I_1 - I_2)$$

Arithmetic Operations of Complex Numbers

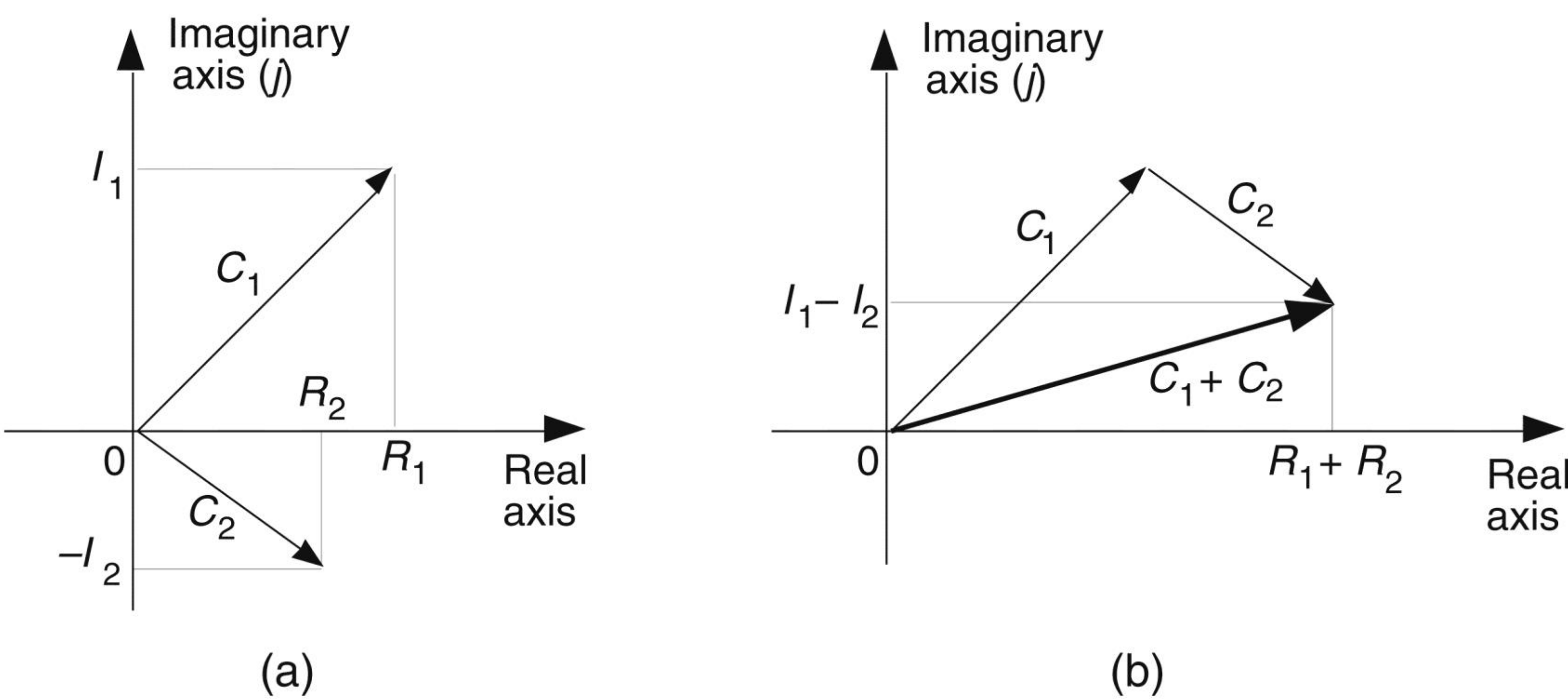

Figure A-3 Geometrical representation of the sum of two complex numbers.

Arithmetic Operations of Complex Numbers

- Multiplication
  - Can use rectangular form to multiply

$$C_1C_2 = (R_1 + jI_1)(R_2 + jI_2) = (R_1R_2 - I_1I_2) + j(R_1I_2 + R_2I_1)$$

  - In exponential form, product takes simpler form

$$C_1C_2 = M_1e^{j\phi_1}M_2e^{j\phi_2} = M_1M_2e^{j(\phi_1+\phi_2)}$$

  - Product of magnitudes of two complex numbers

$$|C_1|\cdot|C_2| = |C_1C_2|$$

  - Scaling

$$kC = k(R + jI) = kR + jkI$$

$$= k(Me^{j\phi}) = kMe^{j\phi}$$

Arithmetic Operations of Complex Numbers

- Conjugation
  - Complex conjugate of a complex number is obtained by changing sign of its imaginary part

$$C = R + jI = Me^{j\phi} \rightarrow C^* = R - jI = Me^{-j\phi}$$

  - Conjugate of a product = product of conjugates

$$C = C_1 C_2$$

$$C^* = (C_1 C_2)^* = (M_1 M_2 e^{j(\phi_1 + \phi_2)})^* = M_1 M_2 e^{-j(\phi_1 + \phi_2)}$$

$$= M_1 e^{-j\phi_1} M_2 e^{-j\phi_2} = C_1^* C_2^*$$

  - Sum of conjugates = conjugate of the sum

$$(R_1 + jI_1)^* + (R_2 + jI_2)^* = (R_1 - jI_1) + (R_2 - jI_2)$$

$$= R_1 + R_2 - j(I_1 + I_2) = [R_1 + R_2 + j(I_1 + I_2)]^*$$

Arithmetic Operations of Complex Numbers

- Conjugation
  - Product of a complex number and its conjugate is complex number's magnitude squared

$$CC^{*} = Me^{j\phi}Me^{-j\phi} = M^{2}e^{j0} = M^{2}$$

Arithmetic Operations of Complex Numbers

- Division

$$\frac{C_1}{C_2} = \frac{M_1 e^{j\phi_1}}{M_2 e^{j\phi_2}} = \frac{M_1}{M_2} e^{j(\phi_1 - \phi_2)}$$

$$\frac{C_1}{C_2} = \frac{M_1}{M_2} \angle(\phi_1 - \phi_2)$$

$$\frac{C_1}{C_2} = \frac{R_1 + jI_1}{R_2 + jI_2}$$

$$= \frac{R_1 + jI_1}{R_2 + jI_2} \cdot \frac{R_2 - jI_2}{R_2 - jI_2}$$

$$= \frac{(R_1 R_2 + I_1 I_2) + j(R_2 I_1 - R_1 I_2)}{R_2^{\ 2} + I_2^{\ 2}}$$

Arithmetic Operations of Complex Numbers

- Inverse of a complex number

$$\frac{1}{C} = \frac{1}{Me^{j\phi}} = \frac{1}{M} e^{-j\phi}$$

$$\frac{1}{C} = \frac{1}{R + jI} \cdot \frac{R - jI}{R - jI} = \frac{R - jI}{R^2 + I^2} = \frac{C^*}{M^2}$$

Arithmetic Operations of Complex Numbers

- Complex numbers raised to a power

$$C = Me^{j\phi} \rightarrow C^k = M^k (e^{j\phi})^k = M^k e^{jk\phi}$$

Arithmetic Operations of Complex Numbers

- Roots of a complex number

$$C = Me^{j\phi} = Me^{j(\phi + n360^\circ)}$$

$$\sqrt[k]{C} = \sqrt[k]{Me^{j(\phi + n360^\circ)}} = \sqrt[k]{M}\, e^{j(\phi + n360^\circ)/k}$$

  - Next, we assign values 0, 1, 2, 3, ..., *k*–1 to *n* to get the *k* roots of *C*

Arithmetic Operations of Complex Numbers

- Natural logarithms of a complex number

$$C = Me^{j\phi}$$

$$\ln C = \ln(Me^{j\phi}) = \ln M + \ln(e^{j\phi}) = \ln M + j\phi$$

where $0 \le \phi < 2\pi$

Arithmetic Operations of Complex Numbers

- Logarithm to base 10 of a complex number

$$C = Me^{j\phi}$$

$$\log_{10} C = \log_{10}(Me^{j\phi}) = \log_{10} M + \log_{10}(e^{j\phi}) = \log_{10} M + j\phi \cdot \log_{10}(e)$$
$$\approx \log_{10} M + j(0.43429 \cdot \phi)$$

Arithmetic Operations of Complex Numbers

- Log to base 10 of a complex number using natural logarithms

$$\log_{10}(x) = \frac{\ln(x)}{\ln(10)}$$

$$C = Me^{j\phi}$$

$$\log_{10} C = \frac{\ln C}{\ln 10} = (\log_{10} e)(\ln C)$$

$$\approx 0.43429 \cdot (\ln C) = 0.43429 \cdot (\ln M + j\phi)$$

Some Practical Implications of Using Complex Numbers

- Representing numbers in a computer
  - Rectangular form has advantage over polar form
  - Example: represent complex numbers using a four-bit sign-magnitude binary number format
    - Integral numbers ranging from –7 to +7
    - Range of complex numbers covers a square on complex plane (Fig. A-4(a)) using rectangular form
    - If we use four-bit numbers to represent magnitude in polar form, those numbers reside on or within a circle whose radius is 7 (Fig. A-4(b))
    - Four shaded corners in Fig. A-4(b) represent locations of valid complex values using rectangular form but are *out of bounds* if we use polar form
      - Acceptable result in rectangular could overflow in polar

Some Practical Implications of Using Complex Numbers

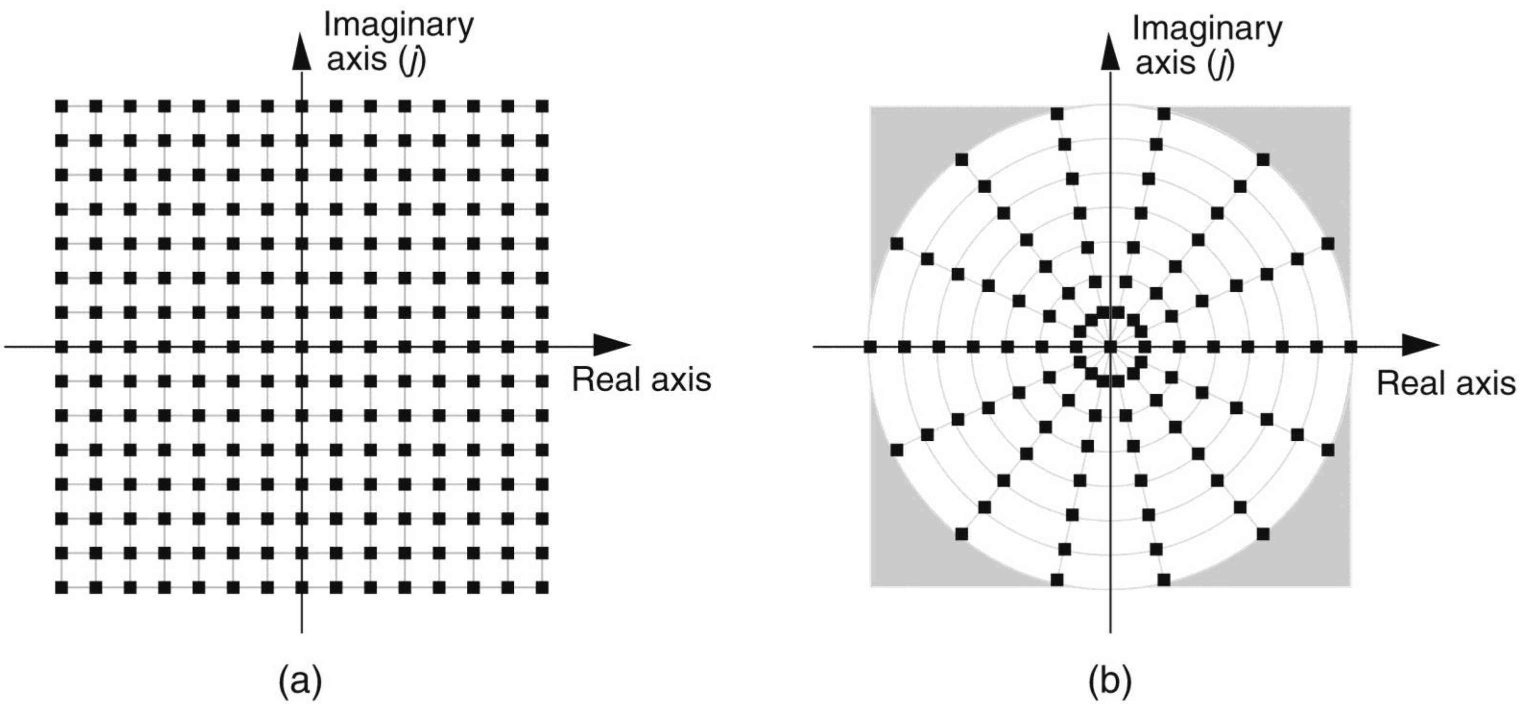

Figure A-4 Complex integral numbers represented as points on the complex plane using a four-bit sign-magnitude data format: (a) using rectangular notation; (b) using polar notation.