J003 - Fundamental Concepts of Computer Science

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Exercise sheet 1

deadline: **17.4.2015**

Common definitions

$$\Sigma_{bool} = \{0, 1\}$$

Definition 2.64 [Theoretical Computer Science, p.44]
A *word* $x \in (\Sigma_{bool})^*$ is said to be *random*, if

$$K(x) \geq |x|$$

A *positive integer* n is said to be *random*, if

$$K(n) = K(Bin(n)) \geq \lceil \log_2(n+1) \rceil - 1$$

Exercise 1.
Prove that, for every $i \in \mathbb{N}$, the interval $[2^i, 2^{i+1} - 1]$ contains at least one random integer.

Exercise 2.
Let $w_n = 0^{2^{n^2}}$ for all $n \in \mathbb{N}$. Give the best possible upper bound on the Kolmogorov complexity of w_n, measured in the length of w_n. (You do not have to prove the optimality of your bound.)

Exercise 3.
Let $L \subseteq (\Sigma_{bool})^*$ be an infinite recursive language with the property that, for any length $k \in \mathbb{N}$, L contains exactly one word w_k. How can the Kolmogorov complexity of w_k be bounded from above?

Exercise 4.
Define an infinite sequence of natural numbers $y_1, y_2, y_3, \ldots$ with $y_i < y_{i+1}$ such that there exists a constant c where, for all $i \geq 1$,

$$K(y_i) \leq \lceil \log_2 \log_2 \log_2 y_i \rceil + c$$

Exercise 5.
Prove that, for every $n \in \mathbb{N}$ and every $i < n$, the interval $[2^n, 2^{n+1} - 1]$ contains at least $2^n - 2^{n-i}$ different numbers x such that $K(x) \geq n - i$.

Exercise 6.
Prove that the following languages are not regular, using the method of Kolmogorov complexity.

(a) $L_1 = \{ww^R \mid w \in \{0,1\}^*\}$, where w^R denotes the reverse of w. (For $w = a_1a_2 \dots a_n$ the reverse of w is defined as $w^R = a_na_{n-1} \dots a_1$.)

(b) $L_2 = \{0^{n^3} \mid n \in \mathbb{N}\}$.

*** Exercise 7.**
Prove that there are at most finitely many prime numbers that can be viewed as random numbers.

*** Exercise 8.**
We consider the language

$$L_{kol} = \{w\#x \mid K(w) \leq Number(x); w, x \in \Sigma_{bool}^*\}$$

(a) Prove that L_{kol} is undecidable.

(b) Is L_{kol} recursively enumerable?