

IA169 System Verification and Assurance

Deductive Verification

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Validation and Verification

- A general goal of V&V is to prove correct behaviour of algorithms.

Reminder

- Testing is incomplete.
- Testing can detect errors but cannot prove correctness.

Conclusion

- Need for another different way of verification.

Goal of formal Verification

- The goal is to show that system behaves correctly with the same level of confidence as it is given with a mathematical proof.

Requirements

- Formally precise semantics of system behaviour.
- Formally precise definition of system properties to be shown.

Methods of formal verification

- Deductive verification
- Model checking
- Abstract interpretation

Deductive verification

Program is correct if

- it **terminates** for a valid input and returns **correct** output.
- There is a need to show two parts – partial correctness and termination.

Partial correctness (Correctness, Soundness)

- If the computation terminates for valid input values (i.e. values for which the program is defined) the resulting values are correct.

Termination (Completeness, Convergence)

- If executed on valid input values, the computation always terminates.

Serial programs (sequential)

- Input-output-closed and finite programs.
 - All input values are known prior program execution.
 - All output values are stored in output variables.
- Examples: Quick sort, Greater common divider, ...

General Principle

- Program instructions are viewed as state transformers.
- The goal is to show that the mutual relation of input and output values is as expected or given by the specification.
- I.e. to verify the correctness of procedure of transformation of input values to output values.

State of Computation

- State of computation of a program is given by the value of program counter and values of all variables.

Atomic predicates

- Basic statements about individual states of the computation.
- The validity is deduced purely from the values of variables given by the state of computation.
- Examples of atomic propositions: $(x == 0)$, $(x1 \geq y3)$.
- Beware of the scope of variables.

Set of States

- Can be described with a Boolean combination of atomic predicates.
- Example: $(x == m) \wedge (y > 0)$

Assertion

- For a given program location defines a Boolean expression that should be satisfied with the current values of program variables in the given location.
- Invariant of a program location.

Assertions – Proving Correctness

- Assigning properties to individual locations of Control Flow Graph.
- Robert Floyd: Assigning Meanings to Programs (1967)

Testing

- Assertion violation serves as a test oracle.

Run-Time Checking

- Checking location invariants during run-time.
- Improved error localisation as the assertion violated relates to a particular program line.

Undetected Errors

- If the error does not exhibit for given input data.
- If the program behaves non-deterministically (parallelism).

Hoare Proof System

Principle

- Programs = State Transformers.
- Specification = Relation between input and output state of computation.

Hoare logic

- Designed for showing partial correctness of programs.
- Let P and Q be predicates and S be a program, then

$$\{P\} S \{Q\}$$

is the so called *Hoare triple*.

Intended meaning of $\{P\} S \{Q\}$

- S is a program that transforms any state satisfying *pre-condition* P to a state satisfying *post-condition* Q .

Example

- $\{z = 5\} x = z * 2 \{x > 0\}$
- Valid triple, though post-condition could be more precise (stronger).
- Example of a stronger post-condition: $\{x > 5 \wedge x < 20\}$.
- Obviously, $\{x > 5 \wedge x < 20\} \implies \{x > 0\}$.

The Weakest Pre-Condition

- P is **the weakest pre-condition**, if and only if
- $\{P\}S\{Q\}$ is a valid triple and
- $\forall P'$ such that $\{P'\}S\{Q\}$ is valid, $P' \implies P$.
- Edsger W. Dijkstra (1975)

How to prove $\{P\} S \{Q\}$

- Pick suitable conditions P' and Q'

- Decomposition into three sub-problems:

$$\{P'\} S \{Q'\} \quad P \implies P' \quad Q' \implies Q$$

- Use axioms and rules of Hoare system to prove $\{P'\} S \{Q'\}$.
- $P \implies P'$ and $Q' \implies Q$ are called proof obligations.
- Proof obligations are **proven in the standard way**.

Axiom

- Assignment axiom: $\{\phi[x \text{ replaced with } k]\} x := k \{\phi\}$

Meaning

- Triple $\{P\}x := y\{Q\}$ is an axiom in Hoare system, if it holds that P is equal to Q in which all occurrences of x has been replaced with y .

Examples

- $\{y+7>42\} x:=y+7 \{x>42\}$ is an axiom
- $\{r=2\} r:=r+1 \{r=3\}$ is not an axiom
- $\{r+1=3\} r:=r+1 \{r=3\}$ is an axiom

Example

- Prove that the following program returns value greater than zero if executed for value of 5.
- Program: $out := in * 2$

Proof

- 1) We built a Hoare triple:
 $\{in = 5\} out := in * 2 \{out > 0\}$
- 2) We deduce/guess a suitable pre-condition:
 $\{in * 2 > 0\}$
- 3) We prove Hoare triple:
 $\{in * 2 > 0\} out := in * 2 \{out > 0\}$ (axiom)
- 4) We prove auxiliary statement:
 $(in = 5) \implies (in * 2 > 0)$

Rule

- Sequential composition:
$$\frac{\{\phi\}S_1\{\chi\} \wedge \{\chi\}S_2\{\psi\}}{\{\phi\}S_1;S_2\{\psi\}}$$

Meaning

- If S_1 transforms a state satisfying ϕ to a state satisfying χ and S_2 transforms a state satisfying χ to a state satisfying ψ then the sequence $S_1; S_2$ transforms a state satisfying ϕ to a state satisfying ψ .

In the proof

- Should $\{\phi\}S_1; S_2\{\psi\}$ be used in the proof, an intermediate condition χ has to be found, and $\{\phi\}S_1\{\chi\}$ and $\{\chi\}S_2\{\psi\}$ have to be proven.

Hoare System – Partial Correctness

Axiom for **skip**: $\{\phi\} \text{ skip } \{\phi\}$

Axiom for **:=**: $\{\phi[x := k]\} x := k \{\phi\}$

Composition rule:
$$\frac{\{\phi\} S_1 \{\chi\} \wedge \{\chi\} S_2 \{\psi\}}{\{\phi\} S_1; S_2 \{\psi\}}$$

Conditional rule:
$$\frac{\{\phi \wedge B\} S_1 \{\psi\} \wedge \{\phi \wedge \neg B\} S_2 \{\psi\}}{\{\phi\} \text{if } B \text{ then } S_1 \text{ else } S_2 \text{ fi} \{\psi\}}$$

While rule:
$$\frac{\{\phi \wedge B\} S \{\phi\}}{\{\phi\} \text{while } B \text{ do } S \text{ od } \{\phi \wedge \neg B\}}$$

Consequence rule:
$$\frac{\phi \implies \phi', \{\phi'\} S \{\psi'\}, \psi' \implies \psi}{\{\phi\} S \{\psi\}}$$

Prove that for $n \geq 0$ the following code computes $n!$.



```
r = 1;
```

```
while (n  $\neq$  0) {  
    r = r * n;
```

```
    n = n - 1;
```

```
}
```

Notes:

Prove that for $n \geq 0$ the following code computes $n!$.

- $\{ n \geq 0 \wedge t=n \}$ {P}
 $r = 1;$

 while ($n \neq 0$) {
 $r = r * n;$

 $n = n - 1;$
 }
 $\{ r=t! \}$ {Q}

Notes:

- Reformulation in terms of Hoare logic.
- Note the use of auxiliary variable t .

Prove that for $n \geq 0$ the following code computes $n!$.

- $\{ n \geq 0 \wedge t=n \}$ {P}
 $r = 1;$
 $\{ n \geq 0 \wedge t=n \wedge r = 1 \}$ {I₁}
 while ($n \neq 0$) {
 $r = r * n;$

 $n = n - 1;$
 }
 $\{ r=t! \}$ {Q}

Notes:

- $\{ n \geq 0 \wedge t=n \wedge 1=1 \} r=1 \{ n \geq 0 \wedge t=n \wedge r=1 \}$
- $(n \geq 0 \wedge t=n) \implies (n \geq 0 \wedge t=n \wedge 1=1)$

Prove that for $n \geq 0$ the following code computes $n!$.

- $\{ n \geq 0 \wedge t=n \}$ {P}
 $r = 1;$
 $\{ n \geq 0 \wedge t=n \wedge r = 1 \}$ {I₁}
 while ($n \neq 0$) $\{ r=t!/n! \wedge t \geq n \geq 0 \}$ {I₂}
 $r = r * n;$

 $n = n - 1;$
 }
 $\{ r=t! \}$ {Q}

Notes:

- Invariant of a cycle: $\{I_2\} \equiv \{ r=t!/n! \wedge t \geq n \geq 0 \}$
- $I_1 \implies I_2$ $(I_2 \wedge \neg(n \neq 0)) \implies Q$

Prove that for $n \geq 0$ the following code computes $n!$.

- $\{ n \geq 0 \wedge t=n \}$ {P}
 $r = 1;$
 $\{ n \geq 0 \wedge t=n \wedge r = 1 \}$ {I₁}
 while ($n \neq 0$) $\{ r=t!/n! \wedge t \geq n \geq 0 \}$ {I₂}
 $r = r * n;$
 $\{ r=t!/(n-1)! \wedge t \geq n > 0 \}$ {I₃}
 $n = n - 1;$
 }
 $\{ r=t! \}$ {Q}

Notes:

- $\{ r*n = t!/(n-1)! \wedge t \geq n > 0 \} r=r*n \{I_3\}$
- $I_2 \wedge (n \neq 0) \implies (r*n = t!/(n-1)! \wedge t \geq n > 0)$

Prove that for $n \geq 0$ the following code computes $n!$.

- $\{ n \geq 0 \wedge t=n \}$ {P}
 $r = 1;$
 $\{ n \geq 0 \wedge t=n \wedge r = 1 \}$ {I₁}
 while ($n \neq 0$) $\{ r=t!/n! \wedge t \geq n \geq 0 \}$ {I₂}
 $r = r * n;$
 $\{ r=t!/(n-1)! \wedge t \geq n > 0 \}$ {I₃}
 $n = n - 1;$
 }
 $\{ r=t! \}$ {Q}

Notes:

- $\{ r = t!/(n-1)! \wedge t \geq (n-1) \geq 0 \} n=n-1$ {I₂}
- $I_3 \implies (r = t!/(n-1)! \wedge t \geq (n-1) \geq 0)$

Observation

- Hoare logic allowed us to reduce the problem of proving program correctness to a problem of proving a set of mathematical statements with arithmetic operations.

Notice about correctness's and (in)completeness

- Hoare logic is correct, i.e. if it is possible to deduce $\{P\}S\{Q\}$ then executing program S from a state satisfying P may terminate only in a state satisfying Q .
- If a proof system is strong enough to express integral arithmetics, it is necessarily incomplete, i.e. there exists claims that cannot be proven or dis-proven using the system.
- Hoare system for proving correctness of programs is incomplete due to the proof obligations generated with the consequence rule.

Troubles with Proof Construction

- Often pre- and post- condition must be suitable reformulated for the purpose of the proof.
- It is very difficult to identify loop invariants.

Partial Correctness in Practice

- Often reduced to formulation of all the loop invariants, and demonstration that they actually are the loop invariants.
- The proof of being an invariant is often achieved with math induction.

Well-Founded Domain

- Partially ordered set that does not contain infinitely decreasing sequence of members.
- Examples: $(\mathbf{N}, <)$, $(PowerSet(\mathbf{N}), \subseteq)$

Proving Termination

- For every loop in the program a suitable well-founded domain and an expression over the domain is chosen.
- It is shown that the value associated with a location cannot grow along any instruction that is part of the loop.
- It is shown that there exists at least one instruction in the loop that decreases the value of the expression.

Automating Deductive Verification

Pre-processing

- Transformation of program to a suitable intermediate language.
- Examples of IL: Boogie (Microsoft Research), Why3 (INRIA)

Structural Analysis and Construction of the Proof Skeleton

- Identification of Hoare triples, loop invariants and suitable pre- and post-conditions (some of that might be given with the program to be verified).
- Generation of auxiliary proof obligations.

Solving proof obligations

- Using tools for automated proving.
- May be human-assisted.

Tools for Automated Proving

- User guides a tool to construct a proof.
- HOL, ACL2, Isabelle, PVS, Coq, ...

Reduced to the satisfiability problem

- Employ SAT and SMT solvers.
- Z3, ...

Proof

- A finite sequence of steps that using axioms and rules of a given proof system that transforms assumptions ψ into the conclusion φ .

Observation

- For systems with finitely many axioms and rules, proofs may be systematically generated. Hence, for all provable claims the proof can be found in finite time.
- All reasonable proving systems has infinitely many axioms. Consider, e.g. an axiom $x = x$. This is virtually a shortcut (template) for axioms $1 = 1$, $2 = 2$, $3 = 3$, etc.
- Semi-decidable with dove-tailing approach.

Searching for a Proof of Valid Statement

- The number of possible finite sequences of steps of rules and axiom applications is too many (infinitely many).
- In general there is no algorithm to find a proof in a given proof system even for a valid statement.
- Without some clever strategy, it cannot be expected that a tool for automated proof generation will succeed in a reasonable short time.
- The strategy is typically given by an experienced user of the automated proving tool. The user typically has to have appropriate mathematical feeling and education.
- At the end, the tool is used as a mechanical checker for a human constructed proof.

Theorem Provery

- The goal is find the proof within a given proof system.
- the proof is searched for in two modes:
 - Algorithmic mode – Application of rules and axioms
 - Guided by the user of the tool.
 - Application of the general proving techniques, such as deduction, resolution, unification,
 - Search mode – Looking for new valid statements
 - Employs brute-force approach and various heuristics.

Existing Tools

- The description of system (axioms, rules) as well as the claim to be proven is given in the language of the tool.

Possible Outputs

- a) Proof has been found and checked.
- b) Proof has not been found.
 - The statement is valid, can be proven, but the proof has not yet been found.
 - The statement is valid, but it cannot be proven in the system.
 - The statement is invalid.

Observation

- In the case that no proof has been found, there is no indication of why it is so.

`http://rise4fun.com/dafny`

Practicals

- Dafny (rise4fun.com)
- http://www.doc.ic.ac.uk/~scd/Dafny_Material/Exercises_%20Arithm_Terms_Loops.docx

Homework

- Prove correctness of the following program using Dafny

```
method Count(N: nat, M: int, P: int) returns (R: int) {
  var a := M;
  var b := P;
  var i := 1;
  while (i <= N) {
    a := a+3;
    b := 2*a+b+1;
    i := i+1;
  }
  R := b;
}
```

- Read and repeat: **Jaco van de Pol**: *Automated verification of Nested DFS*