Numerical features

- Throughout this lecture we assume that all features are numerical, i.e. feature vectors belong to $\mathbb{R}^n$.
- Most non-numerical features can be conveniently transformed to numerical ones.

  For example:

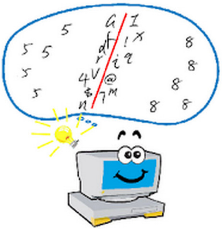
  - Colors $\{\mathit{blue}, \mathit{red}, \mathit{yellow}\}$ can be represented by $\{0, 1, 2\}$ (or $\{-1, 0, 1\}$, ...)
  - A black-and-white picture of $x \times y$ pixels can be encoded as a vector of xy numbers that capture the shades of gray of the pixels.

Basic Problems

We consider two basic problems:

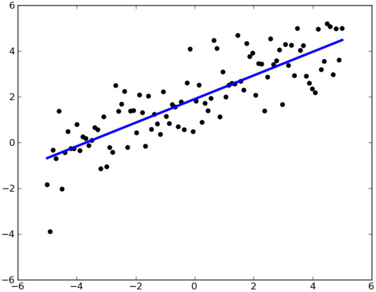
- (Binary) classification

  Our goal: Classify inputs into two categories.



- Function approximation (regression)

  Our goal: Find a (hypothesized) functional dependency in data.



Binary classification in $\mathbb{R}^n$

- Assume
  - a set of instances $X \subseteq \mathbb{R}^n$,
  - an *unknown* categorization function $c : X \to \{0,1\}$.
- **Our goal:**
  - Given a set D of training examples of the form $(\vec{x}, c(\vec{x}))$ where $\vec{x} \in X$,
  - construct a hypothesized categorization function $h \in \mathcal{H}$ that is consistent with c on the training examples, i.e.,

$$h(\vec{x}) = c(\vec{x}) \text{ for all training examples } (\vec{x}, c(\vec{x})) \in D$$

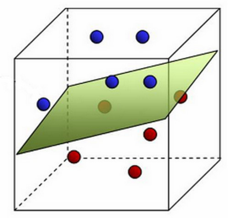
Comments:

- In practice, we often do not strictly demand $h(\vec{x}) = c(\vec{x})$ for all training examples $(\vec{x}, c(\vec{x})) \in D$ (often it is impossible)
- We are more interested in good **generalization**, that is how well h classifies new instances that do not belong to D.
  - Recall that we usually evaluate accuracy of the resulting hypothesized function h on a test set.

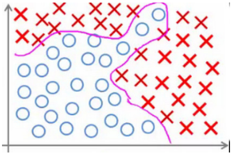
Hypothesis Spaces

We consider two kinds of hypothesis spaces:

- Linear (affine) classifiers (this lecture)



- Classifiers based on combinations of linear and sigmoidal functions (classical neural networks) (next lecture)



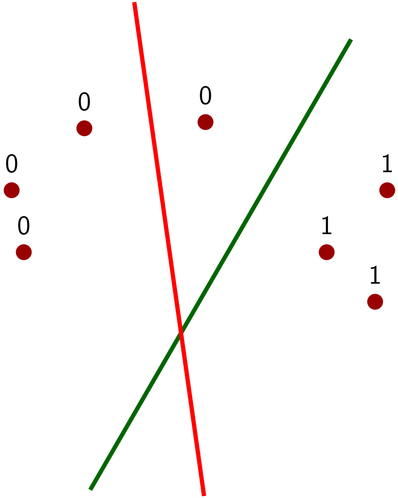
Length and Scalar Product of Vectors

- We consider vectors $\vec{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$.
- Typically, we use Euclidean metric on vectors: $|\vec{x}| = \sqrt{\sum_{i=1}^m x_i^2}$
  The distance between two vectors (points) $\vec{x}, \vec{y}$ is $|\vec{x} - \vec{y}|$.
- We use the *scalar product* $\vec{x} \cdot \vec{y}$ of vectors $\vec{x} = (x_1, \ldots, x_m)$ and $\vec{y} = (y_1, \ldots, y_m)$ defined by

$$\vec{x} \cdot \vec{y} = \sum_{i=1}^{m} x_i y_i$$

  - Recall that $\vec{x} \cdot \vec{y} = |\vec{x}||\vec{y}| \cos \theta$ where θ is the angle between $\vec{x}$ and $\vec{y}$. That is $\vec{x} \cdot \vec{y}$ is the length of the projection of $\vec{y}$ on $\vec{x}$ multiplied by $|\vec{x}|$.
  - Note that $\vec{x} \cdot \vec{x} = |\vec{x}|^2$

Linear classifier - example



- classification in plane using a linear classifier
- if a point is incorrectly classified, the learning algorithm turns the line (hyperplane) to improve the classification.

Linear Classifier

A *linear classifier* $h[\vec{w}]$ is determined by a vector of *weights* $\vec{w} = (w_0, w_1, \ldots, w_n) \in \mathbb{R}^{n+1}$ as follows:

Given $\vec{x} = (x_1, \ldots, x_n) \in X \subseteq \mathbb{R}^n$,

$$h[\vec{w}](\vec{x}) := \begin{cases} 1 & w_0 + \sum_{i=1}^{n} w_i \cdot x_i \geq 0 \\ 0 & w_0 + \sum_{i=1}^{n} w_i \cdot x_i < 0 \end{cases}$$

More succinctly:

$$h(\vec{x}) = sgn\left(w_0 + \sum_{i=1}^{n} w_i \cdot x_i\right) \qquad \text{where} \qquad sgn(y) = \begin{cases} 1 & y \geq 0 \\ 0 & y < 0 \end{cases}$$

Linear Classifier – Notation

Given $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we define an *augmented feature vector*

$$\tilde{\mathbf{x}} = (x_0, x_1, \ldots, x_n) \quad \text{where } x_0 = 1$$

This makes the notation for the linear classifier more succinct:

$$h[\vec{w}](\vec{x}) = sgn(\vec{w} \cdot \tilde{\mathbf{x}})$$

Perceptron Learning

- Given a training set

  $$D = \{(\vec{x}_1, c(\vec{x}_1)), (\vec{x}_2, c(\vec{x}_2)), \dots, (\vec{x}_p, c(\vec{x}_p))\}$$

  Here $\vec{x}_k = (x_{k1} \dots, x_{kn}) \in X \subseteq \mathbb{R}^n$ and $c(\vec{x}_k) \in \{0, 1\}$.

  We write c_k instead of $c(\vec{x}_k)$.
  Note that $\tilde{\mathbf{x}}_k = (x_{k0}, x_{k1} \dots, x_{kn})$ where $x_{k0} = 1$.

- A weight vector $\vec{w} \in \mathbb{R}^{n+1}$ is **consistent with** D if

  $$h[\vec{w}](\vec{x}_k) = sgn(\vec{w} \cdot \tilde{\mathbf{x}}_k) = c_k \quad \text{for all } k = 1, \dots, p$$

  D is **linearly separable** if there is a vector $\vec{w} \in \mathbb{R}^{n+1}$ which is consistent with D.

- Our goal is to find a consistent $\vec{w}$ assuming that D is linearly separable.

Perceptron – Learning Algorithm

Online learning algorithm:

Idea: Cyclically go through the training examples in D and adapt weights. Whenever an example is incorrectly classified, turn the hyperplane so that the example is closer to it's correct half-space.

Compute a sequence of weight vectors $\vec{w}^{(0)}, \vec{w}^{(1)}, \vec{w}^{(2)}, \ldots$.

- $\vec{w}^{(0)}$ is randomly initialized close to $\vec{0} = (0, \ldots, 0)$
- In $(t+1)$-th step, $\vec{w}^{(t+1)}$ is computed as follows:

$$\begin{aligned} \vec{w}^{(t+1)} &= \vec{w}^{(t)} - \varepsilon \cdot \left(h[\vec{w}^{(t)}](\vec{x}_k) - c_k \right) \cdot \tilde{\mathbf{x}}_k \\ &= \vec{w}^{(t)} - \varepsilon \cdot \left(sgn\left(\vec{w}^{(t)} \cdot \tilde{\mathbf{x}}_k \right) - c_k \right) \cdot \tilde{\mathbf{x}}_k \end{aligned}$$

  Here $k = (t \mod p) + 1$, i.e. the examples are considered cyclically, and $0 < \varepsilon \leq 1$ is a **learning speed**.

Věta (Rosenblatt)

If D is linearly separable, then there is t^* such that $\vec{w}^{(t^*)}$ is consistent with D.

Example

Training set:

$$D = \{((2,-1),1),((2,1),1),((1,3),0)\}$$

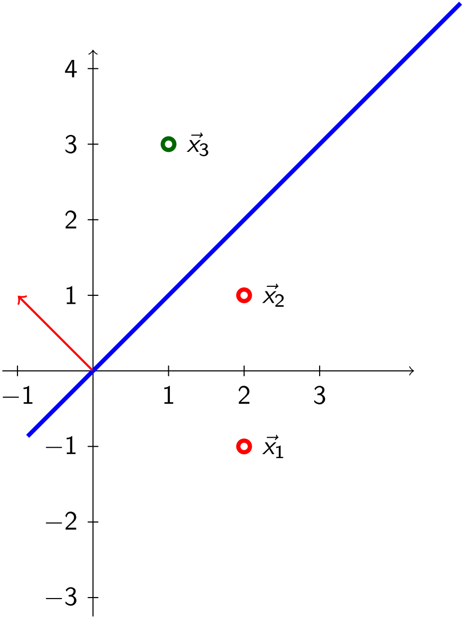
That is

$$\begin{aligned} \vec{x}_1 &= (2,-1) & \qquad \tilde{\mathbf{x}}_1 &= ({\color{red}1},2,-1) \\ \vec{x}_2 &= (2,1) & \tilde{\mathbf{x}}_2 &= ({\color{red}1},2,1) \\ \vec{x}_3 &= (1,3) & \tilde{\mathbf{x}}_3 &= ({\color{red}1},1,3) \end{aligned}$$

$$\begin{aligned} c_1 &= 1 \\ c_2 &= 1 \\ c_3 &= 0 \end{aligned}$$

Assume that the initial vector $\vec{w}^{(0)}$ is $\vec{w}^{(0)} = (0,-1,1)$.
Consider $\varepsilon = 1$.

Example: Separating by $\vec{w}^{(0)}$



Denoting $\vec{w}^{(0)} = (w_0, w_1, w_2) = (0, -1, 1)$ the blue separating line is given by $w_0 + w_1 x_1 + w_2 x_2 = 0$.

The red vector normal to the blue line is (w_1, w_2).

The points on the side of (w_1, w_2) are assigned 1 by the classifier, the others zero.
(In this case $\vec{x}_3$ is assigned one and $\vec{x}_1, \vec{x}_2$ are assigned zero, all of this is inconsistent with $c_1 = 1, c_2 = 1, c_3 = 0$.)

Example: $\vec{w}^{(1)}$

We have

$$\vec{w}^{(0)} \cdot \tilde{\mathbf{x}}_1 = (0, -1, 1) \cdot (1, 2, -1) = 0 - 2 - 1 = -3$$

thus

$$sgn\left(\vec{w}^{(0)} \cdot \tilde{\mathbf{x}}_1\right) = 0$$

and thus

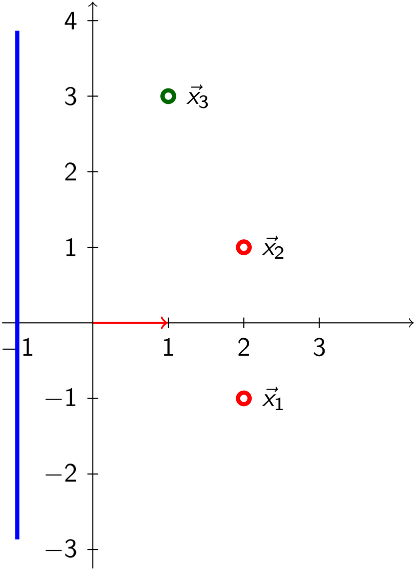
$$sgn\left(\vec{w}^{(0)} \cdot \tilde{\mathbf{x}}_1\right) - c_1 = 0 - 1 = -1$$

(This means that $\vec{x}_1$ is not well classified, and $\vec{w}^{(0)}$ is not consistent with D.)

Hence,

$$\begin{aligned}
\vec{w}^{(1)} &= \vec{w}^{(0)} - \left(sgn\left(\vec{w}^{(0)} \cdot \tilde{\mathbf{x}}_1\right) - c_1\right) \cdot \tilde{\mathbf{x}}_1 \\
&= \vec{w}^{(0)} + \tilde{\mathbf{x}}_1 \\
&= (0, -1, 1) + (1, 2, -1) \\
&= (1, 1, 0)
\end{aligned}$$

Example



Example: Separating by $\vec{w}^{(1)}$

We have

$$\vec{w}^{(1)} \cdot \tilde{\mathbf{x}}_2 = (1,1,0) \cdot (1,2,1) = 1 + 2 = 3$$

thus

$$sgn\left(\vec{w}^{(1)} \cdot \tilde{\mathbf{x}}_2\right) = 1$$

and thus

$$sgn\left(\vec{w}^{(1)} \cdot \tilde{\mathbf{x}}_2\right) - c_2 = 1 - 1 = 0$$

(This means that $\vec{x}_2$ *is* currently well classified by $\vec{w}^{(1)}$. However, as we will see, $\vec{x}_3$ is not well classified.)

Hence,

$$\vec{w}^{(2)} = \vec{w}^{(1)} = (1,1,0)$$

Example: $\vec{w}^{(3)}$

We have

$$\vec{w}^{(2)} \cdot \tilde{\mathbf{x}}_3 = (1,1,0) \cdot (1,1,3) = 1 + 1 = 2$$

thus

$$sgn\left(\vec{w}^{(2)} \cdot \tilde{\mathbf{x}}_3\right) = 1$$

and thus

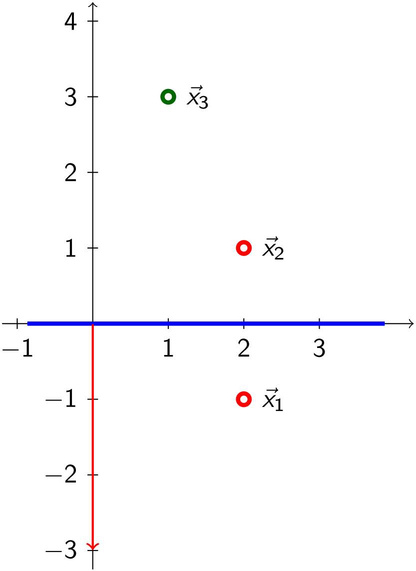
$$sgn\left(\vec{w}^{(2)} \cdot \tilde{\mathbf{x}}_3\right) - c_3 = 1 - 0 = 1$$

(This means that $\vec{x}_3$ is not well classified, and $\vec{w}^{(2)}$ is not consistent with D.)

Hence,

$$\begin{aligned}
\vec{w}^{(3)} &= \vec{w}^{(2)} - \left(sgn\left(\vec{w}^{(2)} \cdot \tilde{\mathbf{x}}_3\right) - c_3\right) \cdot \tilde{\mathbf{x}}_3 \\
&= \vec{w}^{(2)} - \tilde{\mathbf{x}}_3 \\
&= (1,1,0) - (1,1,3) \\
&= (0,0,-3)
\end{aligned}$$

Example: Separating by $\vec{w}^{(3)}$



Example: $\vec{w}^{(4)}$

We have

$$\vec{w}^{(3)} \cdot \tilde{\mathbf{x}}_1 = (0, 0, -3) \cdot (1, 2, -1) = 3$$

thus

$$sgn\left(\vec{w}^{(3)} \cdot \tilde{\mathbf{x}}_1\right) = 1$$

and thus

$$sgn\left(\vec{w}^{(3)} \cdot \tilde{\mathbf{x}}_1\right) - c_1 = 1 - 1 = 0$$

(This means that $\vec{x}_1$ *is* currently well classified by $\vec{w}^{(3)}$. However, as we will see, $\vec{x}_2$ is not.)

Hence,

$$\vec{w}^{(4)} = \vec{w}^{(3)} = (0, 0, -3)$$

Example: $\vec{w}^{(5)}$

We have

$$\vec{w}^{(4)} \cdot \tilde{\mathbf{x}}_2 = (0, 0, -3) \cdot (1, 2, 1) = -3$$

thus

$$sgn\left(\vec{w}^{(4)} \cdot \tilde{\mathbf{x}}_2\right) = 0$$

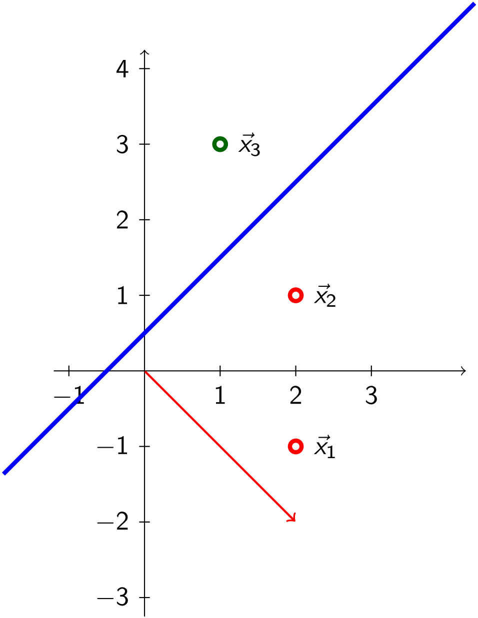
and thus

$$sgn\left(\vec{w}^{(4)} \cdot \tilde{\mathbf{x}}_2\right) - c_2 = 0 - 1 = -1$$

(This means that $\vec{x}_2$ is not well classified, and $\vec{w}^{(4)}$ is not consistent with D.)
Hence,

$$\begin{aligned}
\vec{w}^{(5)} &= \vec{w}^{(4)} - \left(sgn\left(\vec{w}^{(4)} \cdot \tilde{\mathbf{x}}_2\right) - c_2\right) \cdot \tilde{\mathbf{x}}_2 \\
&= \vec{w}^{(4)} + \tilde{\mathbf{x}}_2 \\
&= (0, 0, -3) + (1, 2, 1) \\
&= (1, 2, -2)
\end{aligned}$$

Example: Separating by $\vec{w}^{(5)}$



Example: The result

The vector $\vec{w}^{(5)}$ is consistent with D:

$$sgn\left(\vec{w}^{(5)} \cdot \tilde{\mathsf{x}}_1\right) = sgn\left((1,2,-2)\cdot(1,2,-1)\right) = sgn(7) = 1 = c_1$$

$$sgn\left(\vec{w}^{(5)} \cdot \tilde{\mathsf{x}}_2\right) = sgn\left((1,2,-2)\cdot(1,2,1)\right) = sgn(3) = 1 = c_2$$

$$sgn\left(\vec{w}^{(5)} \cdot \tilde{\mathsf{x}}_3\right) = sgn\left((1,2,-2)\cdot(1,1,3)\right) = sgn(-3) = 0 = c_3$$

Perceptron – Learning Algorithm

Batch learning algorithm:

Compute a sequence of weight vectors $\vec{w}^{(0)}, \vec{w}^{(1)}, \vec{w}^{(2)}, \ldots$.

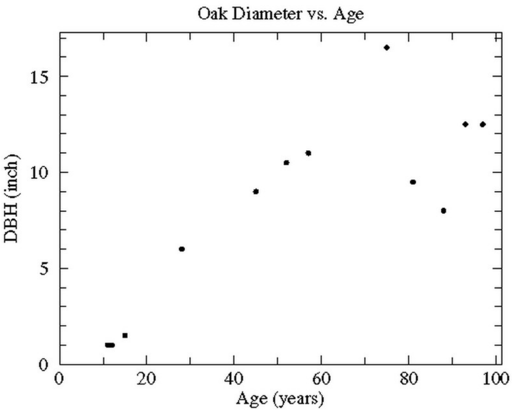
- $\vec{w}^{(0)}$ is randomly initialized close to $\vec{0} = (0, \ldots, 0)$
- In $(t+1)$-th step, $\vec{w}^{(t+1)}$ is computed as follows:

$$\begin{aligned}\vec{w}^{(t+1)} &= \vec{w}^{(t)} - \varepsilon \cdot \sum_{k=1}^{p} \left(h[\vec{w}^{(t)}](\vec{x}_k) - c_k \right) \cdot \tilde{\mathbf{x}}_k \\ &= \vec{w}^{(t)} - \varepsilon \cdot \sum_{k=1}^{p} \left(sgn\left(\vec{w}^{(t)} \cdot \tilde{\mathbf{x}}_k \right) - c_k \right) \cdot \tilde{\mathbf{x}}_k\end{aligned}$$

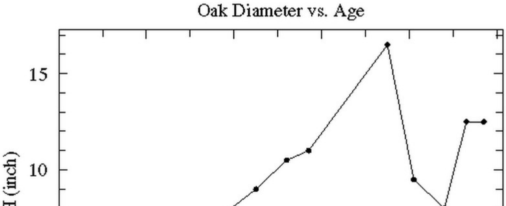
Here $k = (t \mod p) + 1$, i.e. the examples are considered cyclically, and $0 < \varepsilon \leq 1$ is a **learning speed**.

Function Approximation – Oaks in Wisconsin

This example is from *How to Lie with Statistics* by Darrell Huff (1954)



Age (years)	*DBH* (inch)
97	12.5
93	12.5



Function Approximation

- Assume
  - a set $X \subseteq \mathbb{R}^n$ of instances,
  - an *unknown* function $f : X \to \mathbb{R}$.
- **Our goal:**
  - Given a set D of training examples of the form $(\vec{x}, f(\vec{x}))$ where $\vec{x} \in X$,
  - construct a hypothesized function $h \in \mathcal{H}$ such that

    $$h(\vec{x}) \approx f(\vec{x}) \text{ for all training examples } (\vec{x}, f(\vec{x})) \in D$$

    Here $\approx$ means that the values are somewhat close to each other w.r.t. an appropriate *error function* E.
- In what follows we use the *least squares* defined by

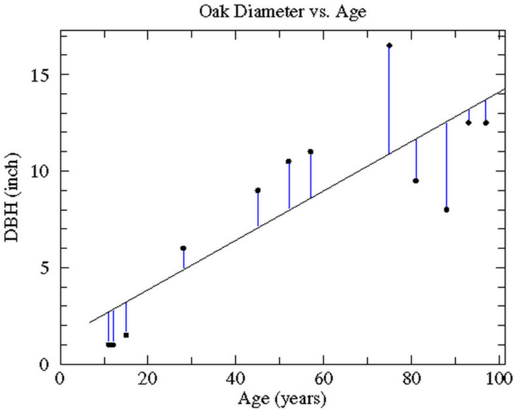
$$E = \frac{1}{2} \sum_{(\vec{x}, f(\vec{x})) \in D} (f(\vec{x}) - h(\vec{x}))^2$$

Our goal is to minimize E.

The main reason is that this function has nice mathematical properties (as opposed e.g. to $\sum_{(\vec{x}, f(\vec{x})) \in D} |f(\vec{x}) - h(\vec{x})|$).

Least Squares – Oaks in Wisconsin

Age (years)	***DBH* (inch)**
97	12.5
93	12.5
88	8.0
81	9.5
75	16.5
57	11.0
52	10.5
45	9.0
28	6.0
15	1.5
12	1.0
11	1.0



Linear Function Approximation

- Given a set D of training examples:

$$D = \{(\vec{x}_1, f(\vec{x}_1)), (\vec{x}_2, f(\vec{x}_2)), \ldots, (\vec{x}_p, f(\vec{x}_p))\}$$

Here $\vec{x}_k = (x_{k1} \ldots, x_{kn}) \in \mathbb{R}^n$ and $f_k(\vec{x}) \in \mathbb{R}$.
Recall that $\widetilde{\mathbf{x}}_k = (x_{k0}, x_{k1} \ldots, x_{kn})$.

Our goal: Find $\vec{w}$ so that $h[\vec{w}](\vec{x}) = \vec{w} \cdot \widetilde{\mathbf{x}}$ approximates the function f some of whose values are given by the training set.

- **Least Squares Error Function:**

$$E(\vec{w}) \;=\; \frac{1}{2}\sum_{k=1}^{p}\left(\vec{w} \cdot \widetilde{\mathbf{x}}_k - f_k\right)^2 \;=\; \frac{1}{2}\sum_{k=1}^{p}\left(\sum_{i=0}^{n} w_i x_{ki} - f_k\right)^2$$

Gradient of the Error Function

Consider the **gradient** of the error function:

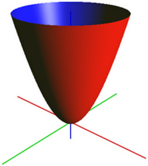
$$\nabla E(\vec{w}) = \left(\frac{\partial E}{\partial w_0}(\vec{w}), \ldots, \frac{\partial E}{\partial w_n}(\vec{w}) \right) = \sum_{k=1}^{p} (\vec{w} \cdot \tilde{\mathbf{x}}_k - f_k) \cdot \tilde{\mathbf{x}}_k$$

What is the gradient $\nabla E(\vec{w})$? It is a vector in $\mathbb{R}^{n+1}$ which points in the direction of the steepest *ascent* of E (it's length corresponds to the steepness). Note that here the vectors $\tilde{\mathbf{x}}_k$ are *fixed* parameters of E!

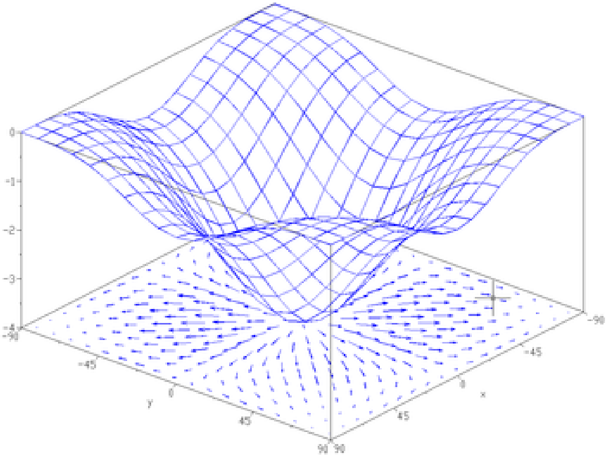
Fakt

If $\nabla E(\vec{w}) = \vec{0} = (0, \ldots, 0)$, *then* $\vec{w}$ *is a global minimum of* E.

This follows from the fact that E is a convex paraboloid that has a unique extreme which is a minimum.



Gradient – illustration



Function Approximation – Learning

Gradient Descent:

- Weights $\vec{w}^{(0)}$ are initialized randomly close to $\vec{0}$.
- In $(t+1)$-th step, $\vec{w}^{(t+1)}$ is computed as follows:

$$\begin{aligned}
\vec{w}^{(t+1)} &= \vec{w}^{(t)} - \varepsilon \cdot \nabla E(\vec{w}^{(t)}) \\
&= \vec{w}^{(t)} - \varepsilon \cdot \sum_{k=1}^{p} \left(\vec{w}^{(t)} \cdot \tilde{\mathbf{x}}_k - f_k \right) \cdot \tilde{\mathbf{x}}_k \\
&= \vec{w}^{(t)} - \varepsilon \cdot \sum_{k=1}^{p} \left(h[\vec{w}^{(t)}](\vec{x}_k) - f_k \right) \cdot \tilde{\mathbf{x}}_k
\end{aligned}$$

Here $k = (t \mod p) + 1$ and $0 < \varepsilon \leq 1$ is the learning speed.

Note that the algorithm is almost similar to the batch perceptron algorithm!

Tvrzení

For sufficiently small $\varepsilon > 0$ the sequence $\vec{w}^{(0)}, \vec{w}^{(1)}, \vec{w}^{(2)}, \ldots$ converges (component-wisely) to the global minimum of E.

Finding the Minimum in Dimension One

Assume $n = 1$. Then the error function E is

$$E(w_0, w_1) = \frac{1}{2}\sum_{k=1}^{p}(w_0 + w_1 x_k - f_k)^2$$

Minimize E w.r.t. w_0 a w_1:

$$\frac{\delta E}{\delta w_0} = 0 \quad \Leftrightarrow \quad w_0 = \bar{f} - w_1\bar{x} \quad \Leftrightarrow \quad \bar{f} = w_0 + w_1\bar{x}$$

where $\bar{x} = \frac{1}{p}\sum_{k=1}^{p} x_k$ a $\bar{f} = \frac{1}{p}\sum_{k=1}^{p} f_k$

$$\frac{\delta E}{\delta w_1} = 0 \quad \Leftrightarrow \quad w_1 = \frac{\frac{1}{p}\sum_{k=1}^{p}(f_k - \bar{f})(x_k - \bar{x})}{\frac{1}{p}\sum_{k=1}^{p}(x_k - \bar{x})^2}$$

i.e. $w_1 = cov(f, x)/var(x)$

Finding the Minimum in Arbitrary Dimension

Let A be a matrix $p \times (n+1)$ (p rows, $n+1$ columns) whose k-th row is the vector $\tilde{\mathbf{x}}_k$.

Let $\vec{f} = (f_1, \ldots, f_p)^\top$ be the *column* vector formed by values of f in the training set.

Then

$$\nabla E(\vec{w}) = 0 \quad \Leftrightarrow \quad \vec{w} = (A^\top A)^{-1} A^\top \vec{f}$$

if $(A^\top A)^{-1}$ exists

(Then $(A^\top A)^{-1} A^\top$ is the so called Moore-Penrose pseudoinverse of A.)

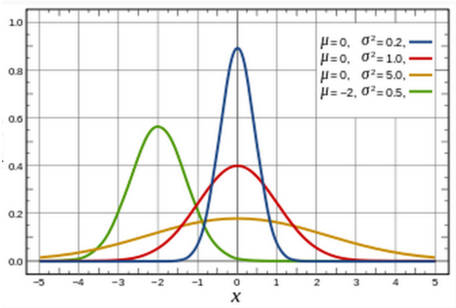
Normal Distribution – Reminder

Distribution of continuous random variables.

Density (one dimensional, that is over $\mathbb{R}$):

$$p(x) \quad = \quad \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} \quad =: \quad N[\mu,\sigma^2](x)$$

μ is the expected value (the mean), σ^2 is the variance.



Maximum Likelihood vs Least Squares (Dim 1)

Fix a training set $D = \{(x_1, f_1), (x_2, f_2), \ldots, (x_p, f_p)\}$
Assume that each f_k has been generated randomly by

$$f_k = (w_0 + w_1 \cdot x_k) + \epsilon_k$$

Here

- w_0, w_1 are **unknown numbers**
- ϵ_k are normally distributed with mean 0 and an unknown variance σ^2

Assume that $\epsilon_1, \ldots, \epsilon_p$ were generated **independently**.

Denote by $p(f_1, \ldots, f_p \mid w_0, w_1, \sigma^2)$ the probability density according to which the values $f_1, \ldots, f_n$ were generated assuming fixed $w_0, w_1, \sigma^2, x_1, \ldots, x_p$.

(For interested: The independence and normality imply

$$p(f_1, \ldots, f_p \mid w_0, w_1, \sigma^2) = \prod_{k=1}^{p} N[w_0 + w_1 x_k, \sigma^2](f_k)$$

where $N[w_0 + w_1 x_k, \sigma^2](f_k)$ is a normal distribution with the mean $w_0 + w_1 x_k$ and the variance σ^2.)

Maximum Likelihood vs Least Squares

Our goal is to find (w_0, w_1) that maximizes the likelihood that the training set D with **fixed** values $f_1, \ldots, f_n$ has been generated:

$$L(w_0, w_1, \sigma^2) := p(f_1, \ldots, f_p \mid w_0, w_1, \sigma^2)$$

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(w_0, w_1) *maximizes* $L(w_0, w_1, \sigma^2)$ *for arbitrary* σ^2 **iff** (w_0, w_1) *minimizes* $E(w_0, w_1)$*, i.e. the least squares error function.*

Note that the maximizing/minimizing (w_0, w_1) does not depend on σ^2.

Maximizing σ^2 satisfies $\sigma^2 = \frac{1}{p}\sum_{k=1}^{p}(f_k - w_0 - w_1 \cdot x_k)^2$.

Comments on Linear Models

- Linear models are parametric, i.e. they have a fixed form with a small number of parameters that need to be learned from data (as opposed e.g. to decision trees where the structure is not fixed in advance).
- Linear models are stable, i.e. small variations in the training data have only limited impact on the learned model. (tree models typically vary more with the training data).
- Linear models are less likely to overfit (low variance) the training data but sometimes tend to underfit (high bias).