Tutorial III

14 March 2019 12:02

TAIL INEQUALITIES

Markov's inequality

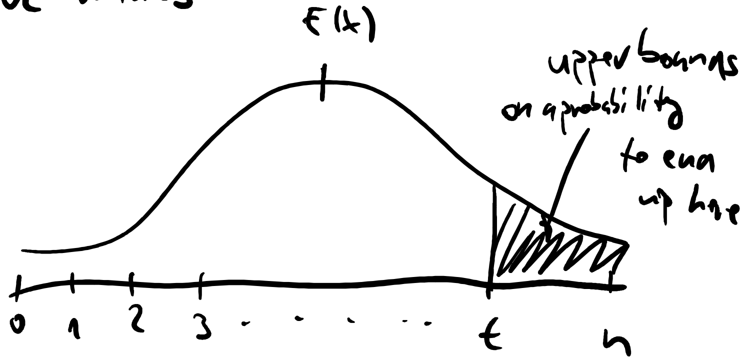
Chebyshev's inequality + exercise + probability amplification

Chernoff's inequality

MARKOV'S INEQUALITY

X - a random variable with positive values

$$\Pr(X \ge t) \le \frac{E(X)}{t}$$



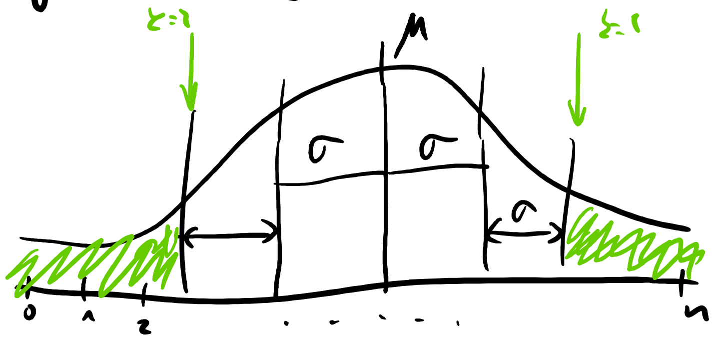
Useful for $t > E(X)$

Chebyshev's inequality

X has finite $E(X) = \mu$ $Var(X) = \sigma^2$ σ - standard deviation

$$\Pr(|X - \mu| \ge k\sigma) \le \frac{1}{k^2}$$

$$\Pr(|X - \mu| \ge t) \le \frac{\sigma^2}{t^2}$$



Useful only for $k > 1$

Chernoff's inequalities

Specific form of random variables

$X = \sum_{i=1}^{n} X_i$, where X_i are identically independently distributed binary random variables with

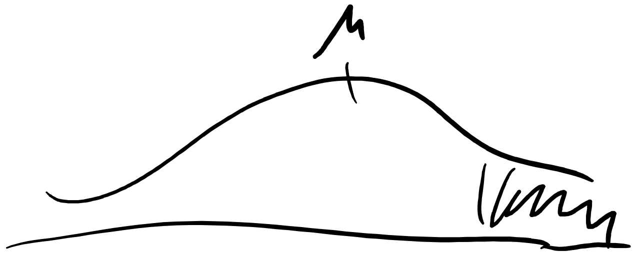
$Pr(X_i = 1) = p \Rightarrow E(X_i) = p \Rightarrow E(X) = n \cdot p$

$E(X) = E(\sum_i X_i) = \sum_i E(X_i) = n \cdot p$

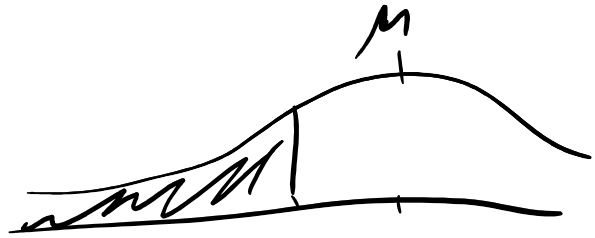
General:

Euler's e

$$Pr(X > (1+\delta)\mu) < \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu}$$



$$Pr(X < (1-\delta)\mu) < \left(\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}}\right)^{\mu}$$

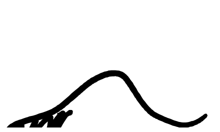


These are inconvenient to work with.

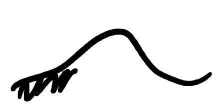
Simpler (but looser) expressions are:

$$Pr(X \leq (1-\delta)\mu) \leq e^{-\frac{\delta^2 \mu}{2}} \qquad 0 \leq \delta \leq 1$$

$$Pr(X \geq (1+\delta)\mu) \leq e^{-\frac{\delta^2 \mu}{3}} \qquad 0 \leq \delta \leq 1$$



$$\Pr(X \ge (1+\delta)\mu) \le e^{-\frac{\delta^2 \mu}{3}} \qquad 0 \le \delta \le 1$$



Also (similar to Chebyshev's inequality)

$$\Pr(|X-\mu| \ge \delta\mu) \le 2e^{-\mu\delta^2/3}$$

Prove using

$$\Pr\big((X-\mu \ge \delta\mu) \vee (\mu - X \ge \delta\mu)\big)$$

$$= \Pr(X-\mu \ge \delta\mu) + \Pr(\mu - X \ge \delta\mu)$$

$$= \Pr(X \ge \delta\mu + \mu) + \Pr(X \le -\delta\mu + \mu)$$

$$= \Pr(X \ge (1+\delta)\mu) + \Pr(X \le (1-\delta)\mu)$$

$$\le e^{-\frac{\delta^2\mu}{3}} + e^{-\frac{\delta^2\mu}{2}} \le 2e^{-\delta^2\mu/3}$$

Exercise

We have a 6-sided die. Let X be the number of times that outcome '6' appears over n throws of the die.

Let p be the probability that $X \ge n/4$

Find bounds on p using above bounds.

$$p = \sum_{i \ge n/4}^{n} \binom{n}{i} p^i (1-p)^{n-i}$$

1.) Markov's inequality

$$E(X) = n/6$$

$$P\left(X \geq \frac{n}{4}\right) \leq \frac{E(X)}{n/4} = \frac{n/6}{n/4} = 2/3$$

2.) Chebyshev's inequality

$$E(X) = n/6 \quad Var(X) = n \cdot p \cdot (1-p) = n \cdot \frac{1}{6} \cdot \frac{5}{6} = \frac{5n}{36}$$

$$P_r\left(X \geq \frac{n}{4}\right)$$

$$P_r\left(|X - E(X)| \geq t\right) \leq \frac{Var(X)}{t^2}$$

$$P_r\left(X - E(X) \geq t\right) \leq P_r\left(|X - E(X)| \geq t\right) = P_r\left(\left|X - \frac{n}{6}\right| \geq \frac{n}{12}\right) \leq \frac{\frac{5n}{36}}{\frac{n^2}{144}} = \frac{20}{n}$$

$$\|$$

$$P_r\left(X \geq E(X) + t\right)$$

$$\|$$

$$P_r\left(X \geq n/4\right)$$

$$t + \frac{n}{6} = \frac{n}{4}$$

$$t = \frac{n}{12}$$

Chernoff's inequality

$$\Pr(X \geq (1+\delta)\mu) \leq e^{-\frac{\delta^2 \mu}{3}} \qquad \Pr\left(X \geq (1+1/2)\frac{n}{6}\right) \leq e^{-\frac{\frac{1}{2}\cdot\frac{n}{6}}{3}}$$

$$\shortparallel \qquad\qquad\qquad \shortparallel$$

$$\Pr\left(X \geq \frac{n}{4}\right) \qquad\qquad e^{-\frac{n}{72}}$$

$$(1+\delta)\cdot\frac{n}{6} = \frac{n}{4}$$

$$\delta = \frac{1}{2}$$

$(0 < \varepsilon \leq 1/2)$

Amplification of success probability

BPP - Probability of a correct result $\geq 1/2 + \varepsilon$

PP - Probability of a correct result $> 1/2$

X_i - characterizes i^{th} run

$X_i = 1$ if the correct answer was given

$X_i = 0$ otherwise

$$\Pr(X_i = 1) = 1/2 + \varepsilon =: p$$

$$\Pr(X_i = 0) = 1/2 - \varepsilon$$

X ~> number of correct runs

$$X = \sum_i X_i \implies E(X) = n\cdot(1/2 + \varepsilon)$$

Probability of an incorrect outcome of repetition algorithm.

Probability of an incorrect outcome of repetition algorithm.

$$\Pr\left(X \le \frac{n}{2}\right)$$

$$\frac{n}{2} = (1-\delta)\cdot\mu$$

$$= (1-\delta)\cdot\left(\frac{n}{2} + n\cdot\varepsilon\right)$$

$$\Leftrightarrow -\frac{\delta n}{2} + n\varepsilon - \delta n\varepsilon = 0$$

$$\vdots$$

$$\delta = \frac{\varepsilon}{\frac{1}{2}+\varepsilon} = \frac{\varepsilon}{p}$$

$$\delta^2 \quad \mu$$

$$\Pr\left(X \le \left(1-\frac{\varepsilon}{p}\right)\cdot n\cdot p\right) \le e^{-\frac{\frac{\varepsilon^2}{p^2}\cdot n\cdot p}{2}}$$

$$= e^{-\frac{n\varepsilon^2}{2p}} = e^{-\frac{n\varepsilon^2}{1+2\varepsilon}}$$

$$e^{-\frac{n\varepsilon^2}{1+2\varepsilon}} \le \delta \qquad / \ln$$

$$-\frac{n\varepsilon^2}{1+2\varepsilon} \le \ln\delta$$

$$-n\varepsilon^2 \le \ln\delta\cdot(1+2\varepsilon)$$

$$n \ge -\frac{\ln\delta\cdot(1+2\varepsilon)}{\varepsilon^2}$$

let k be the size of the input.
→ if ε does not depend on k, n doesn't depend on k either.

k, n doesn't depend on z either.

$\rightarrow$ let $\varepsilon = \frac{1}{f(k)}$ for a polynomial f.

$\rightarrow$ let $\varepsilon = \frac{1}{2^k}$