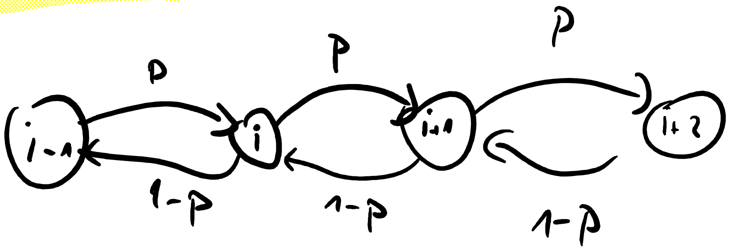
Tutorial IX

25 April 2019 12:05

MARKOV CHAINS II

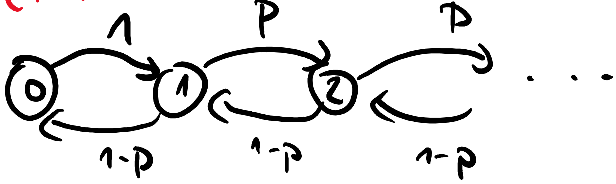
- → Walks on a line
- → 2-SAT
- → Fair 2-colorability of 3-colorable graphs

Walks on a line

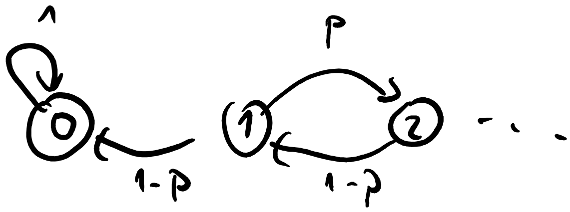


Barriers and their types

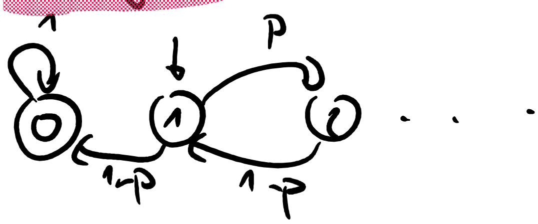
Reflective



Absorbing

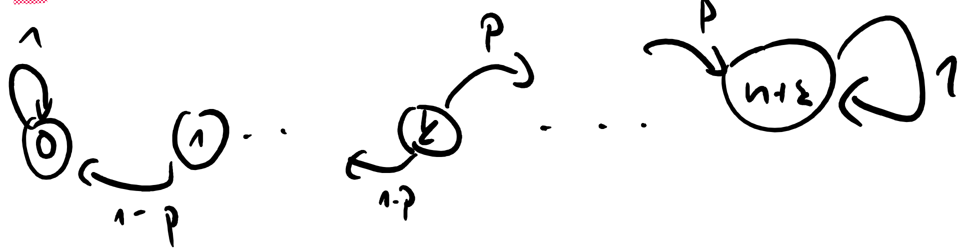


Monkey at a cliff

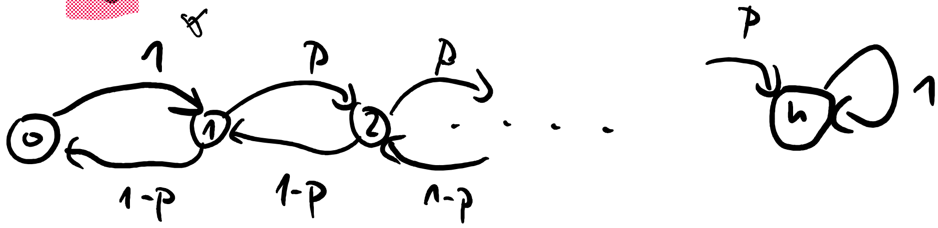


Gambler's ruin

Gamblers ruin



Today we will discuss:



→ What is the average time to reach n from 0,

E_{ij} - expected time to reach j from i

$E_{jj} = 0$

$E_{0,n} = E_{0,1} + E_{1,2} + E_{2,3} + \dots + E_{n-1,n}$

$E_{01} = 1$

$\forall i \; E_{i,i+2} = E_{i,i+1} + E_{i+1,i+2}$ Substitute

$$E_{i,i+1} = 1 + (1-p)E_{i-1,i+1} + pE_{i+1,i+1}$$

$$= 1 + (1-p)\left(E_{i-1,i} + E_{i,i+1}\right)$$

$$= 1 + (1-p)E_{i-1,i} + (1-p)E_{i,i+1}$$

$$pE_{i,i+1} = 1 + (1-p)E_{i-1,i}$$

$$E_{i,i+1} = \frac{1}{p} + \frac{(1-p)}{p}E_{i-1,i}$$

$E_{i,i+1} = r_i$

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$$\boxed{\begin{aligned} r_0 &= 1 \\ r_i &= 1/p + \frac{(1-p)}{p} r_{i-1} \end{aligned}}$$

Solution is easy to calculate: http://www.cs.unipr.it/purrs/

2-SAT problem

Logical formula with variables $x_1 \dots x_n$

$$(x_1 \lor x_2) \land (x_3 \lor \neg x_2) \land (x_4 \lor x_1) \land \dots \land \overbrace{(\neg x_1 \lor \neg x_5)}^{\text{term}}$$

is this satisfiable?

if yes there exists an assignment A which satisfies all the terms.

Randomized algorithm to find A if it exists

1.) Assign values to variables at random

2.) Find an unsatisfied term (if it doesn't exist you found the solution

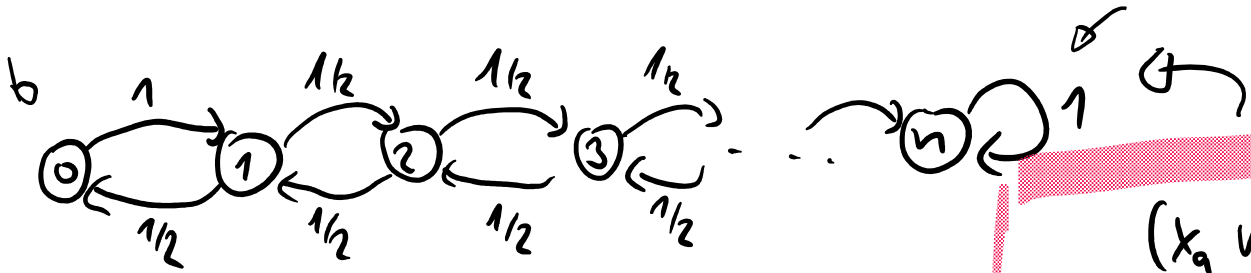
with variables x_a and x_b. Randomly pick x_a or x_b

and flip it's assignment. GO TO 2.)

This is a Las Vegas algorithm. If A exists it finds it (in some number of steps), if it doesn't you can stop a procedure in fixed time and say 'I don't know'.

How long to find the solution? Use walk on a line.

We count the number of variables in an intermediate solution with assignments identical to A.



$(x_a \vee x_b)$

at least one of x_a, x_b has a wrong assignment.

$\Rightarrow$ w.p. $> 1/2$ number of correct assignments increases

the upper bound on the average number of steps to find A is $E_{0,n}$.

$E_{0,1} = 1$

$E_{i,i+1} = \frac{1}{p} + \frac{(1-p)}{p} E_{i-1,i}$

$p = 1/2$

$\boxed{\text{Solution}}$

$r_0 = 1 \qquad E_{i,i+1} = 2i+1$

$r_i = 2 + r_{i-1}$

$$E_{0,n} = E_{0,1} + E_{1,2} + \dots + E_{n-1,n}$$

$$= \sum^{n-1} 2i+1$$

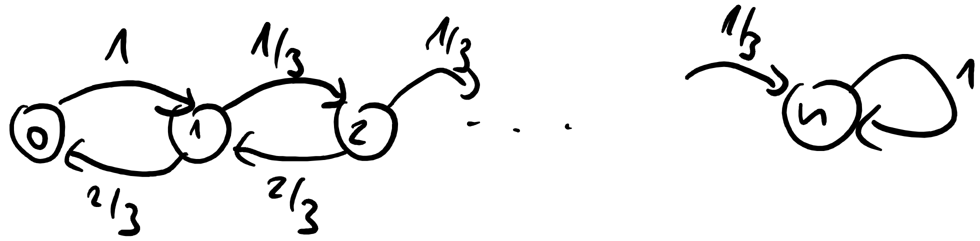
$$= \sum_{i=0} 2i+1$$

$$= n + 2 \cdot \sum_{i=1}^{n-1} i = n + 2\left(\frac{n\cdot(n-1)}{2}\right) = n^2$$

The complete algorithm

Run the procedure $2\cdot n^2$ times. If the solution is not found say 'I don't know'.

Why doesn't this work for 3-SAT?



$(x_a \vee x_b \vee x_c)$

$E_{0,1} = 1$

$E_{i,i+1} = 1/p + \frac{(1-p)}{p} \cdot E_{i-1,i}$

for $p = 1/3$

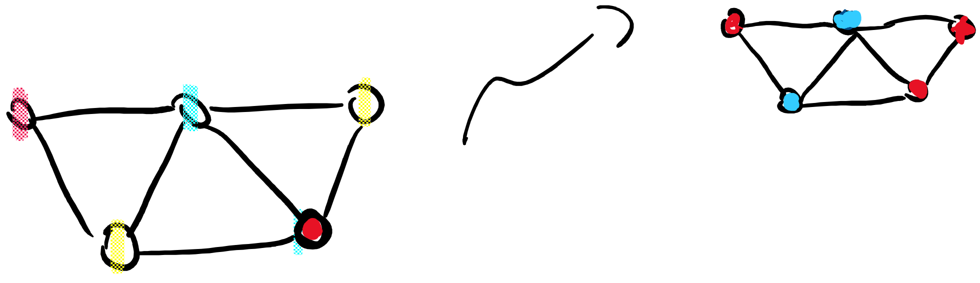
$r_0 = 1$ $\overset{\text{Solution}}{\Longrightarrow}$ $r_i = 4 \cdot 2^i - 3$

$r_i = 3 + 2 \cdot r_{i-1}$

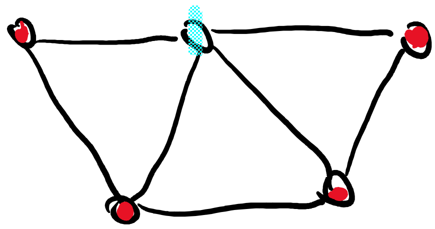
Problem 6.5 from Randomized Algorithms book

let G be a 3-colorable graph

Let G be a 3-colorable graph



Task: Find 2-coloring of G, such that there is no monochromatic triangle.



3 coloring → 2 coloring : Choose one of three colors and change it to one of the remaining colors.

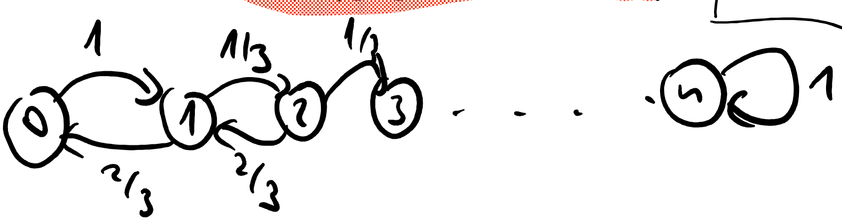
Randomized procedure

1.) Choose a random 2-coloring.

2.) Find a monochromatic triangle T. — if no such triangle exists you have a solution.

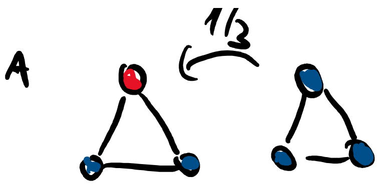
3.) Choose one of vertices of T and change it's color.

ANALYSIS I (WRONG) – Choose the correct coloring A and count the number of colors in intermediate coloring equal to A



$E_i = 4\,2^i - 3i$ (this is slow) A

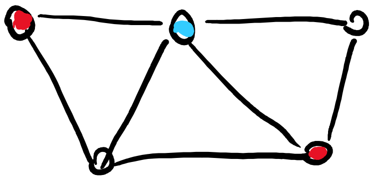
$E_{i,i+1} = 4\cdot 2^{i} - 3$! (this is slow)



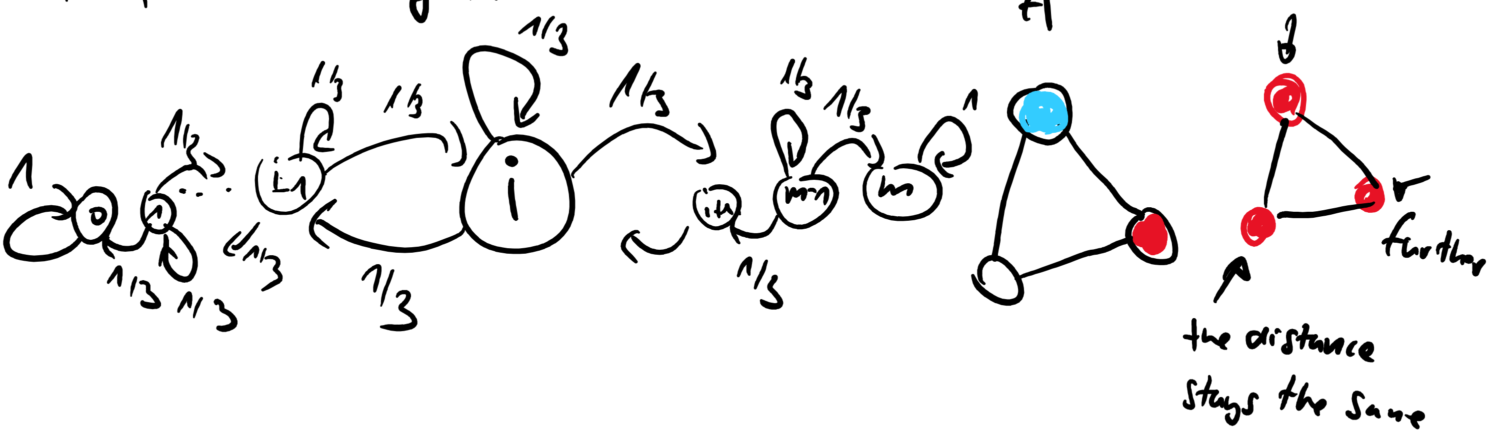
ANALYSIS II

Consider a partial 2-coloring A, which satisfies fairness (no monocolored triangle) ↳ colors $m < n$ vertices

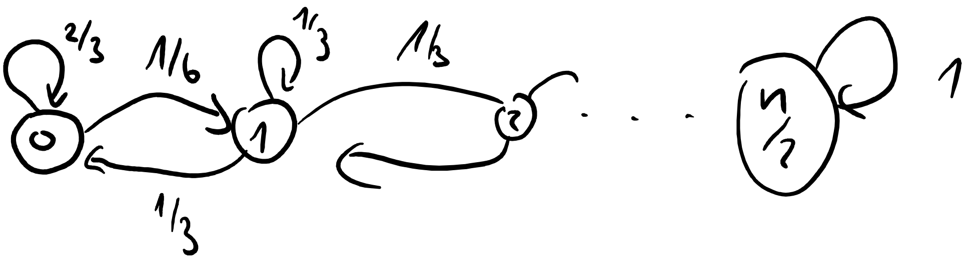
Example



in the Markov chain count the number of colorings equivalent to partial coloring A.



Solution: first reduce to



Then

$E_{01} = \frac{2}{3}E_{01} + 1 \Rightarrow E_{01} = 3$

$E_{i,i+1} = 1 + \frac{1}{3}E_{i-1,i+1} + \frac{1}{3}E_{i,i+1} + \frac{1}{3}\overset{=0}{E_{i,i}}$

$$\frac{2}{3} E_{i,i+1} = 1 + \frac{1}{3}\left(E_{i-1,i} + E_{i,i+1}\right)$$

$$E_{i,i+1} = 3 + E_{i-1,i}$$

Recursion

$r_0 = 3$ Solution $\Rightarrow$ $r_n = 3(n+1)$

$r_i = 3 + r_{i-1}$

$$\sum_{i=0}^{n/2} 3(n+1) = 3\left(\frac{n}{2}+1\right) + 3\sum_{i=0}^{n/2} i = 3\left(\frac{n}{2}+1\right) + 3\left(\frac{\left(n/2+1\right)\cdot\frac{n}{2}}{2}\right)$$

$$= \frac{3}{2}n + 3 + \frac{3}{2}\left(\frac{n^2}{4} + \frac{n}{2}\right)$$

$$= \frac{3}{2}n + \frac{3}{4}n + 3 + \frac{3}{8}n^2$$

$$= \frac{9}{4}n + \frac{3}{8}n^2 + 3 \in O(n^2)$$