Tutorial V

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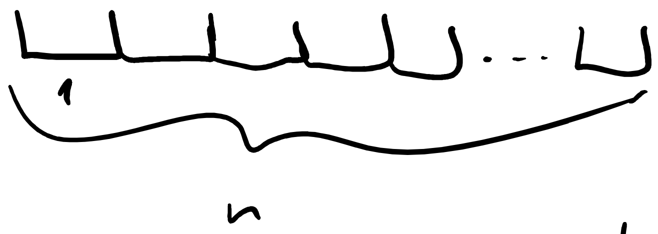
Basic methods: Moments and Deviations

Occupancy problem (bins and balls)

Coupon collector problem

m - balls
n - bins



[bin number 1] - What is the expected number of balls you need to place before one of them lands in bin 1.

Geometric distribution λ

λ_i

1 Probability 1st ball falls into 1st bin $= \frac{1}{n}$

2 Probability 2nd ball is the first one to land into bin 1

$\frac{1}{n} \cdot \frac{n-1}{n}$

⋮

n Probability that n-th ball is the first one to land into bin 1

u) Probability that n-th ball is the first one to land into bin 1

$E(X) = \frac{1}{p}$ if the probability success is p.

$$E(X) = \sum_{i=1}^{\infty} i \cdot P(X=i)$$
$$= \sum_{i=1}^{\infty} i \cdot (1-p)^{i-1} \cdot p$$
$$= p \cdot \sum_{i=1}^{\infty} i \cdot (1-p)^{i-1} \qquad / \cdot (1-p)$$

$$(1-p) \cdot E(X) = p \cdot \sum_{i=1}^{\infty} i \cdot (1-p)^{i}$$

$$(1-(1-p))E(X) = p \cdot \sum_{i=1}^{\infty} i \cdot (1-p)^{i-1} - p \cdot \sum_{i=1}^{\infty} i \cdot (1-p)^{i}$$
$$= p\Big[\overbrace{1 - (1-p)}^{i=1} + \overbrace{2 \cdot (1-p) - 2 \cdot (1-p)^2}^{i=2}$$
$$+ \underbrace{3 \cdot (1-p)^2 - 3 \cdot (1-p)^3}_{i=3}$$

$$= p\Big[1 + (1-p) + (1-p)^2 + (1-p)^3 + \dots +$$

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1 \dots}$$

$= \dots 1$

$$= p \cdot \frac{1}{1-(1-p)}$$

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$$

$$|r| < 1$$

$$p \cdot E(X) = p \cdot \frac{1}{p}$$

$$E(X) = \frac{1}{p}$$

m- balls and n- bins

What is the expected number of empty bins.

$X_i = 1$ if i^{th} bin is empty

$$P_r(X_i = 1) = \left(\frac{n-1}{n}\right)^m$$

$$E(X_i) = \left(\frac{n-1}{n}\right)^m$$

$$X = \sum_i X_i$$

$$E(X) = \sum_{i=1}^{n} E(X_i) = n \cdot \left(\frac{n-1}{n}\right)^m$$

DRUNKEN SAILOR PROBLEM

Coupon collector problem

I have 0 coupons ⇝ Probability that the next coupon I get is new?

$Pr = 1$

I have 1 coupon ⇝ $\underset{(2nd)}{Pr}(New) = \frac{n-1}{n}$

2 coupon ⇝ $\underset{(3rd)}{Pr}(new) = \frac{n-2}{n}$

i^{th} coupon ⇝ $\underset{(i+1)st}{Pr}(new) = \frac{n-i+1}{n}$

these are called "E_m"s in the slides

X_i ⇝ the average number of coupons to get $i+1$ st coupon

these are geometric distributions with $p_i = \left(\frac{n-i+1}{n}\right)$

$$X = \sum_i X_i$$

$$E(X) = E\left(\sum_i X_i\right) = \sum_i E(X_i)$$

$$= \sum_{i=1}^{n} \frac{1}{p_i} = \sum_{i=1}^{n} \frac{n}{n-i} = n \cdot \sum_{i=1}^{n} \frac{1}{n-i+1}$$

$$= n \cdot \sum_{i=1}^{n} \frac{1}{i} = n \cdot H_n \approx n \cdot \log n$$

$\mathcal{E}_j(k)$ = event that j^{th} bin has k or more balls

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n-balls and n-bins

$$\Pr(j\text{th bin has } \underline{\text{exactly}}\ i \text{ balls}) = \binom{n}{i}\left(\frac{1}{n}\right)^i \underbrace{\left(1-\frac{1}{n}\right)^{n-i}}$$

$$\leq \binom{n}{i}\left(\frac{1}{n}\right)^i$$

$$\leq \left(\frac{ne}{i}\right)^i\left(\frac{1}{n}\right)^i \qquad \left(\binom{n}{i} \leq \left(\frac{ne}{i}\right)^i\right)$$

$$= \left(\frac{e}{i}\right)^i$$

$$\boxed{\Pr \mathcal{E}_j(k) = \sum_{i=k}^{n} \binom{n}{i}\left(\frac{1}{n}\right)^i\left(\frac{n-1}{n}\right)^{n-i}}$$

$$\leq \sum_{i=k}^{n} \left(\frac{e}{i}\right)^i = \left(\frac{e}{k}\right)^k + \left(\frac{e}{k+1}\right)^{k+1} + \left(\frac{e}{k+2}\right)^{k+2} + \dots$$

$$\leq \left(\frac{e}{k}\right)^k + \left(\frac{e}{k}\right)^{k+1} + \left(\frac{e}{k}\right)^{k+2} + \dots$$

$$= \left(\frac{e}{k}\right)^k\left(1 + \left(\frac{e}{k}\right) + \left(\frac{e}{k}\right)^2 + \dots + \left(\frac{e}{k}\right)^n\right)$$

$$\leq \left(\frac{e}{k}\right)^k \sum_{i=0}^{\infty}\left(\frac{e}{k}\right)^i$$

if $\left(\frac{e}{k}\right) < 1$

$$\leq \left(\frac{e}{k}\right)^k \cdot \frac{1}{1-\frac{e}{k}}$$

for $k = \left\lceil \frac{e(\ln n)}{\ln \ln n} \right\rceil$

$$\boxed{\leq \frac{1}{n^2}}$$

| couldn't reach this =)

for $k = \left\lceil \frac{\cdots}{\ln \ln n} \right\rceil$ $\qquad$ $\frac{\cdots}{n^2}$

$$\Pr\left[\bigcup_{i=1}^{n} \mathcal{E}_i(k)\right] \le \sum_{i=1}^{n} \Pr \mathcal{E}_i(k) = \frac{1}{n}$$