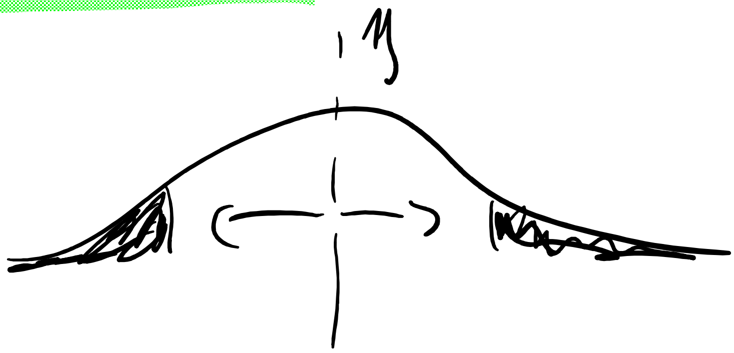
Tutorial VI

04 April 2019 12:01

Chernoff Bounds and Monte-Carlo estimation

Hypercube routing



$X = \sum_{i=1}^{n} X_i$ with X_i being Poisson r.v. $\Pr(X_i = 1) = p$

↳ 0/1 values

$E(X) = n \cdot p = \mu$ X_i are i.i.d.

$$\left.\begin{aligned} &\Pr(X \leq (1-\delta)\cdot\mu) \leq e^{-\frac{\mu\delta^2}{2}} \\ &\Pr(X \geq (1+\delta)\mu) \leq e^{-\frac{\mu\delta^2}{3}} \end{aligned}\right\} \quad 0 \leq \delta \leq 1$$

$$\Pr(|X-\mu| \geq \delta\cdot\mu) \leq 2 \cdot e^{-\mu\delta^2/3}$$

Polling problem

Two presidential candidates A and B. We want to estimate number of people who will vote for A. Assume everyone will vote.

Let's define answer of ith person with r.v. X_i

Such that $X_i = 1$ votes for A

$X_i = 0$ votes for B

$Pr(X_i = 1) = p$ (p is the number we want to estimate)

After asking n people, the estimate $X = \frac{X_1 + X_2 + \dots + X_n}{n}$

$Pr(|X - p| \leq (0.1) \cdot p) > 90\%$

$Pr(|X - p| \geq \frac{p}{10}) \leq 10\%$

good to use Chernoff bound, but doesn't contain n

$E(X) = \sum_{i=1}^{n} E\left(\frac{X_i}{n}\right) = \frac{1}{n} \sum_{i=1}^{n} E(X_i)$

$= \frac{1}{n} \cdot np = p$

$Pr\left(|nX - np| \geq \frac{np}{10}\right) \leq \frac{1}{10}$

$E(nX) = n \cdot p = \mu$

$\leq 2 \cdot e^{-\frac{\mu \cdot \delta^2}{3}}$

$2 \cdot e^{-\frac{n \cdot p \cdot}{100 \cdot 3}} \leq \frac{1}{10}$

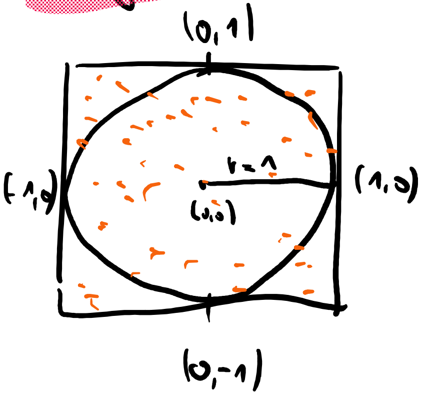
$e^{-\frac{n \cdot p}{300}} \leq \frac{1}{20}$ $/\ln$

$-\frac{n \cdot p}{300} \leq \ln\left(\frac{1}{20}\right)$

$$n \geq -\frac{300}{p} \cdot \ln\left(\frac{1}{20}\right)$$

$$n \geq 900/p$$

Algorithm to estimate π



$z_i = 1$ if ith point falls inside the circle
$ = 0$ if — " — falls outside — " —

$$Pr(z_i = 1) = \frac{V(\text{circle})}{V(\text{square})} = \frac{\pi r^2}{(2r)^2} = \frac{\pi}{4}$$

After n trials we have $z = \sum_i z$

$$E(z) = \sum_{i=1}^{n} E(z) = \frac{n \cdot \pi}{4}$$

$z' = \frac{4 \cdot z}{n}$ is my estimate of π $\quad E(z') = \pi$

$$E(z') = E\left(\frac{4}{n} \cdot z\right) = \frac{4}{n} \cdot E(z) = \pi$$

$$Pr(|z' - \pi| \leq \varepsilon \cdot \pi)$$

$$Pr(|z' - \pi| \geq \varepsilon \cdot \pi) \leq 2 \cdot e^{-\frac{\pi \varepsilon^2}{3}} \qquad /n \qquad \left(\text{in slides } n/4\right)$$

$$\|$$

$$Pr(|nz' - n\pi| > n \cdot \varepsilon \cdot \pi)$$

$E(z') \quad .. \quad E(z')$

$$Pr\left(|nZ' - n\pi| > n\cdot\varepsilon\cdot\pi\right)$$

$$E(nZ') = nE(Z') = n\cdot\pi$$

$$\le 2^{-\frac{n\cdot\pi\cdot\varepsilon^2}{3\cdot(4)}}$$ – in slides

For sphere and cube

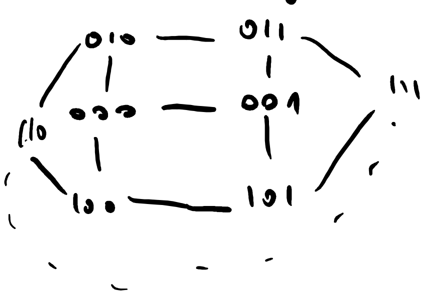
$$Pr(Z_i = 1) = \frac{4/3\,\pi\cdot r^3}{8} = \frac{\pi}{6} \qquad (\pi/6)$$

$$Z' = \sum_i^n Z_i \cdot 6 \qquad E(Z') = \pi$$

$$2^{-\frac{n\pi\varepsilon^2}{3\cdot 6}}$$

Routing in Hypercubes

$N = 2^d$ nodes labeled by binary strings



two nodes are connected if their label differs only in 1 bit.

Total number of edges $\frac{Nd}{2}$ total number of links in our problem is N.d

source node, destination node

Packet: $(i, x, d(i))$

data

Assumption 1 packet in a link per time

Experiment

Each node gets a packet with a different destination

How long will it take to deliver all of them?

node

$x_1 \dots x_n \to x_n \dots x_1$

i 11|00 $d(i) = 0011$
j 01|00 $d(j) = 0010$
k 10|00 $d(k) = 0001$

$x_1 \dots x_n \overbrace{0\,0 \dots}^{n} \rightsquigarrow \overbrace{0\,0\,0}^{n}\; x_n \dots x_1$

Left to right routing

i: 1100 0100 0000 0010 0011

j: 0100 0000 0010

k: 1000 0000 0001

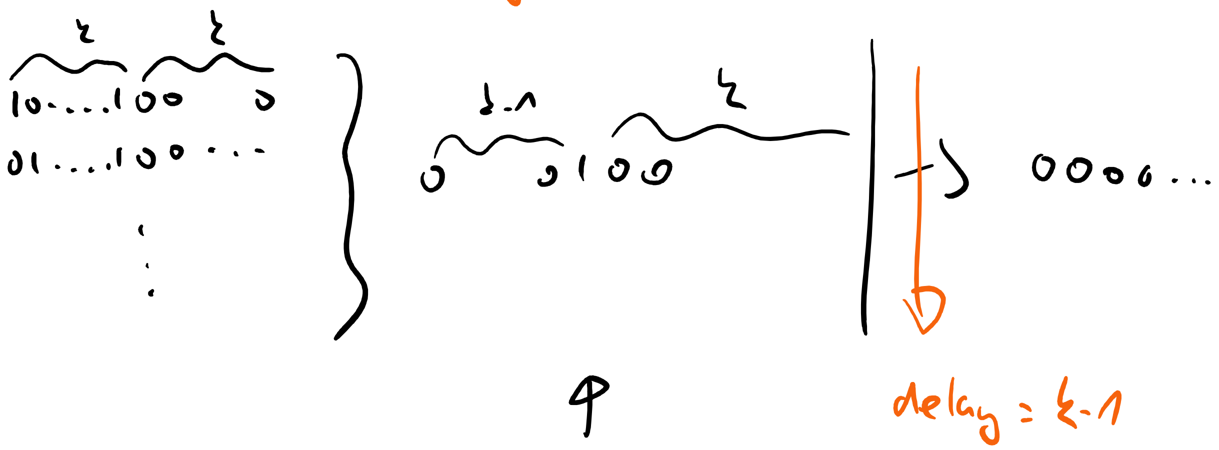
i: 101000 000101

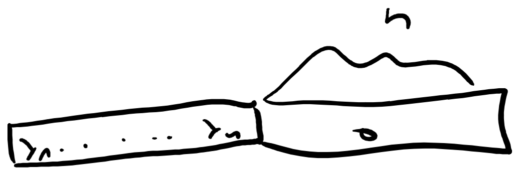
j: 011000 000110

101000 ⇝ 001000 | → 000000 ⇝ 000100 ⇝ 000101

011000 ⇝ 001000 | ⇝ 000000 ⇝ 000100 ⇝ 000110

delay





$$\Omega\left(\frac{2^n}{n}\right)$$

$(i, X, d(i))$ goes to randomly chosen intermediate node $\sigma(i)$

only afterwards to $d(i)$

$$i \nearrow \sigma(i) \searrow d(i)$$