Tutorial VII

11 April 2019 12:03

PROBABILISTIC METHOD

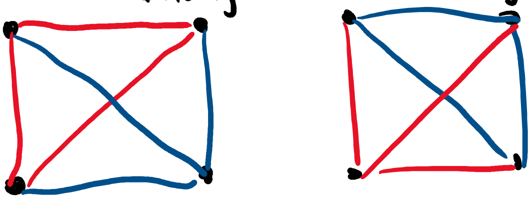
Ramsey number

Ramsey number $R(k,t)$.

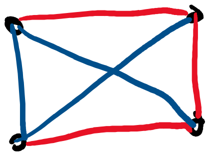
is the smallest number of vertices of a complete graph K_n, such that each two coloring of edges of K_n has a red subgraph K_k or blue subgraph K_t.

R(3,3)

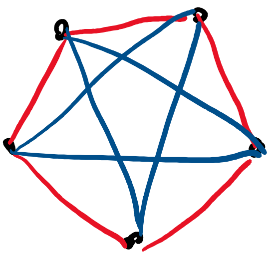
no blue triangle   no red triangle



How many colorings? $2^6 = 64$

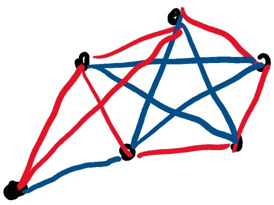
→ 

~> no red or blue triangle $\Rightarrow R(3,3) > 4$

→ 

~> no red or blue triangle $= R(3,3) > 5$

Can we try 6? $2^{\binom{6}{2}}$ colorings $= 2^{15}$ choices!



$R(3,3) = 6$

PROBABILISTIC ARGUMENT

-> Color the graph "randomly" and if probability of a counter example is larger than 0, counter example exists. => lower bounds.

thm from the slides

$$\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1 \quad \Rightarrow R(k,k) > n$$

to show this, let us design a random coloring experiment:

Color every edge of K_n blue w. p. $1/2$, red w. p. $1/2$

Choose $S \subset V$, $|S| = k$

r_S ~ graph induced by S is all red

b_S ~ graph induced by S is all blue

$$Pr(r_S) = 1/2^{\binom{k}{2}}$$

$$Pr(b_S) = 1/2^{\binom{k}{2}}$$

$$Pr(b_S) = 1/2^{\binom{k}{2}}$$

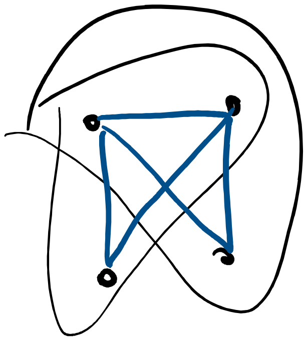
$$Pr(r_S \vee b_S) = 2 \cdot 1/2^{\binom{k}{2}} = 2^{1-\binom{k}{2}}$$ (probability that S is monocolored)

What is the probability that some $X \subset V$, $|X| = k$ is monocolored?

$$Pr\left(\bigvee_{X \subset V, |X| = k} r_X \vee b_X \right)$$ (probability that some monocolored K_k exists)

$$< \underbrace{\binom{n}{k} \cdot 2^{1-\binom{k}{2}}}$$

events are not mutually exclusive



$$1 - P\left(\bigvee_{X \subset V, |X| = k} r_X \vee b_X \right)$$ is a probability that graph contains no monocolored induced subgraph of size k. = COUNTEREXAMPLE

We need $1 - P(\bigvee r_X \vee b_X) > 0$

$$\Downarrow$$

$$P(\bigvee r_X \vee b_X) < 1$$

$$\| $$

$$\binom{n}{k} 2^{1-\binom{k}{2}} < 1$$ => existence of a counter example to a hypothesis that $R(k,k) = n$.

therefore n is a lower bound.

To obtain a good lower bound (the best possible with this method) fix k and find largest n for which $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$.

if $n = \lfloor 2^{\frac{k}{2}} \rfloor$ then $R(k,k) \geq n$

what is $R(2,2) = 2$

Plug $n = \lfloor 2^{\frac{k}{2}} \rfloor$ to the claim and verify if it is true

$$\binom{n}{k} 2^{1-\binom{k}{2}} \leq \frac{n^k}{k!} 2^{1-\frac{k(k-1)}{2}} = \frac{n^k}{k!} \cdot$$

$$= \frac{2^{k/2}}{k!} \cdot \frac{2^{(k-1)}}{2^{k/2}}$$

$$= \frac{2}{k! \; 2^{(k-1)}}$$

$$k!\,2^{(k-1)}$$

for $(k=2)$
$$\frac{2}{2\cdot 2} = \frac{1}{2} < 1$$

and it decreases with k.

What is this bound for $k=3$?

$$n = \lfloor 2^{3/2} \rfloor = \lfloor \sqrt{8} \rfloor = 2$$

$$k = 8$$

$$n = \lfloor 2^{\frac{8}{2}} \rfloor = 16$$

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$R(8,8) > 49$ counterexample

$$R(k,k) > (k-1)^2 \quad \underline{\text{constructive}}$$

$$> 2^{\binom{k}{2}}$$

Show that if

$$\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{t} (1-p)^{\binom{t}{2}} < 1$$

then $\underset{\sim\sim\sim}{R(k,t)} \geq n$

Random experiment: color each edge blue w.p. p
——— " ——— red w.p. $1-p$

Choose $B \subset V$, $|B| = k$

$$\Pr(B \text{ is all blue}) = p^{\binom{k}{2}}$$

$$\Pr(\text{Some subset of size } k \text{ is all blue}) \leq \binom{n}{k} p^{\binom{k}{2}}$$

Choose $R \subset V$, $|R| = t$

$$\Pr(R \text{ is all red}) = (1-p)^{\binom{t}{2}}$$

$$\Pr(\text{Some subset of size } k \text{ is all red}) < \binom{n}{t}(1-p)^{\binom{t}{2}}$$

Probability of a graph consistent with $R(k,t) = n$

$$< \binom{n}{k} p^{\binom{k}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}}$$

if this number is smaller than 1, there is a counterexample with positive probability.

For $R(4,t)$:

$$\binom{n}{4} p^6 + \dots$$

$$p = n^{-2/3}$$

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$$\frac{n^4}{4!} \cdot p^6$$

$$\downarrow$$

$$\frac{1}{24}$$