Tutorial VIII

18 April 2019 11:43

Markov chains

- hitting probability
- hitting time
- stationary distributions

Definition

Markov Chain (MC) is an infinite collection of r.v. $\{X_i\}_{i=0}^{\infty}$ with n outcomes, such that:

$$\forall i \quad Pr(X_i = x_i \mid X_{i-1} = x_{i-1}, X_{i-2} = x_{i-2}, \ldots, X_0 = x_0)$$

$$\parallel$$

$$Pr(X_i = x_i \mid X_{i-1} = x_{i-1})$$

in general process

$$Pr(X_3 = 3 \mid X_2 = 2, X_1 = 1) \neq Pr(X_3 = 3 \mid X_2 = 2, X_1 = 2)$$

INTERPRETATION:
Markov chains are simplest non-trivial stochastic processes. Each X_i is a state of the process in time step i. The process can have n states. Markov property says that the state of the process in step $i+1$ depends only on state in time i and not on the whole past

This leads to simplification:

Probability to move from State 2 to state 3.

$$Pr(X_3 = 3 \mid X_2 = 2) = Pr(X_7 = 3 \mid X_6 = 2) = P_{23}$$

How many probabilities to describe MC? $n \times n$!

Matrix representation = Transition Matrix

$$\begin{pmatrix} P_{11} & P_{12} & P_{13} & \cdots & P_{1n} \end{pmatrix}$$

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1n} \\ p_{21} & p_{22} & & & \\ \vdots & & \ddots & & \\ p_{n1} & & & & p_{nn}\end{pmatrix} \quad \text{(transition matrix)}$$

This matrix is stochastic:

$$\forall_i \sum_j p_{ij} = 1$$

How to calculate probability of the states if we start in l-th state

$$(000\ldots\overset{\text{position } l}{1}\ldots000)\cdot P = \underline{(p_{l1}, p_{l2}, \ldots, p_{ln})}\cdot P = (00\ldots100)\cdot P^2$$

$$= \ldots\ldots$$

in k steps

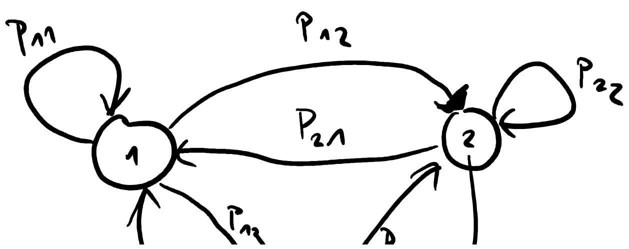
$$(00\ldots\overset{l\text{-th}}{1}000)\, P^k$$

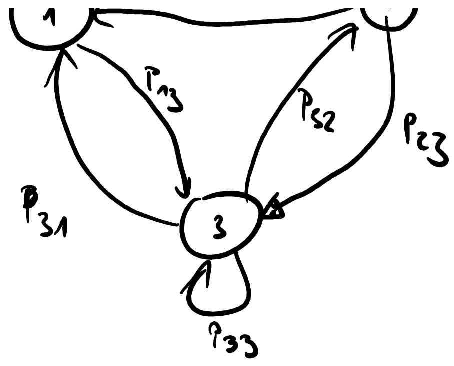
P^k – k step transition matrix

GRAPH REPRESENTATION

Graph with n vertices (vertex corresponds to a state) and directed edges labeled by transition probabilities

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$





Basic properties

$r_{ij}^{(t)}$ → the probability to reach j from i for the first time in exactly t steps.

→ **Hitting probability** – the probability to reach j from i

$$f_{ij} = \sum_{t=0}^{\infty} r_{ij}^{(t)}$$

→ **Hitting time** – average time to reach j from i

$$h_{ij} = \sum_{t=0}^{\infty} t \cdot r_{ij}^{(t)}$$

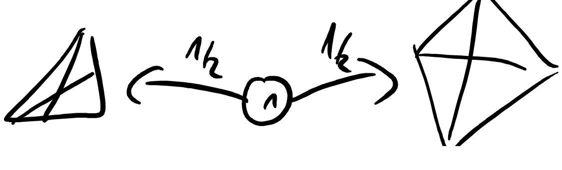
Ergodic theorem

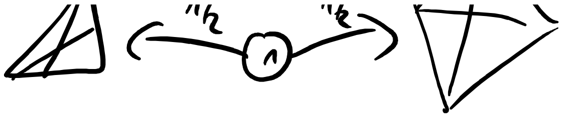
for an ERGODIC MC (all states can reach all other states w.p. 1 and MC is not periodic), there exists a unique probability vector $\Pi = (\Pi_1, \dots, \Pi_n)$ such that

→ $\Pi = \Pi P$

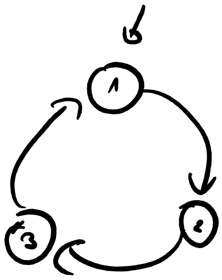
and for all j $\Pi_j = \lim_{t \to \infty} (P_{ij})^t$.

element of P^t matrix



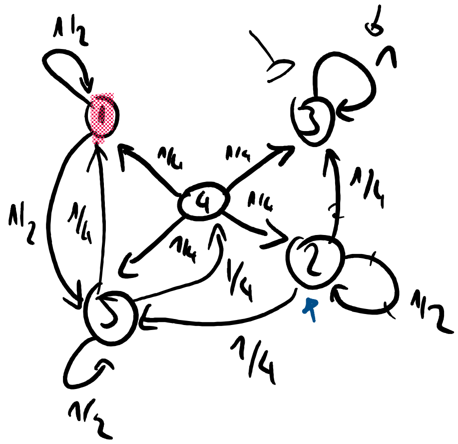


Periodicity



EXERCISES

$$P = \begin{pmatrix} 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/4 & 0 & 1/4 \\ 0 & 0 & 1 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 0 & 1/4 \\ 1/4 & 0 & 0 & 1/4 & 1/2 \end{pmatrix} \rightarrow p_{ij} = \Pr(X_i = j \mid X_{i-1} = i)$$



TASK: calculate f_{i4} for each i

$$f_{44} = 1$$

$$f_{14} = 1/2 \cdot f_{14} + 1/2 \cdot f_{54} \Rightarrow f_{14} = f_{54}$$

$$f_{24} = 1/4 \cdot f_{34} + 1/2 \cdot f_{24} + 1/4 \cdot f_{54}$$

$$f_{34} = 0$$

$$f_{54} = 1/4 \cdot f_{14} + 1/4 \cdot f_{44} + 1/2 f_{54}$$

$$f_{54} = 1/4 f_{54} + 1/4 + 1/2 f_{54}$$

$$1/4 f_{54} = 1/4 \Rightarrow f_{14} = f_{54} = 1$$

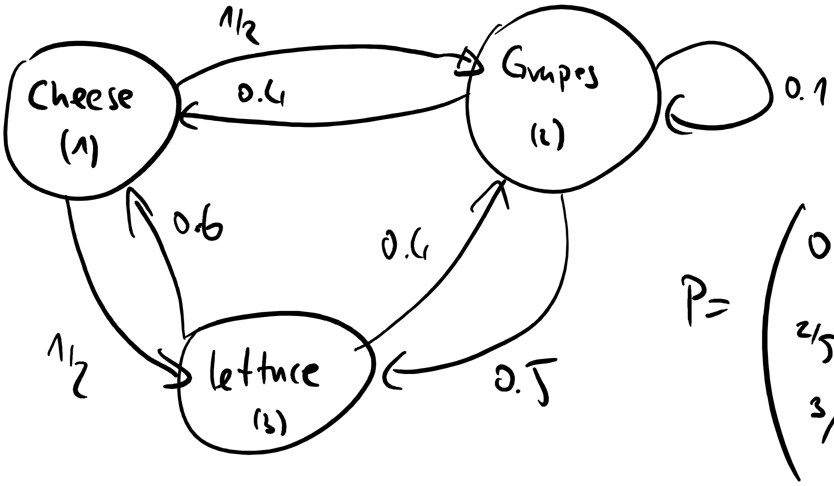
$$f_{24} = 1/4 \cdot 0 + 1/2 f_{24} + 1/4$$

$$1/2 f_{24} = 1/4$$

$$f_{24} = 1/2$$

$f_{24} = 1/2$

$f_{i4} = (1, 1/2, 0, 1, 1)$



$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 2/5 & 1/10 & 1/2 \\ 3/5 & 2/5 & 0 \end{pmatrix}$$

Calculate f_{i3}

$f_{33} = 1$

$f_{23} = 0.5 \cdot f_{33} + 0.1 f_{23} + 0.4 f_{13}$

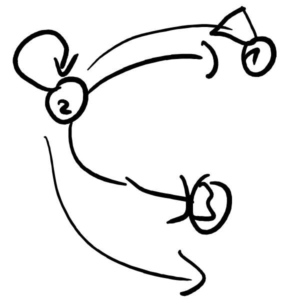
$f_{13} = (0.5) f_{33} + (0.5) \cdot f_{23} \Rightarrow f_{13} = \frac{1}{2} f_{23} + \frac{1}{2}$

$f_{13} = f_{23}$

$f_{23} = 0.5 + 0.1 (f_{23}) + (0.4) f_{23} \Rightarrow f_{23} = 1$

$f_{ij} = 1$

Calculate h_{i3} (average time to hit state 3 from all other states).



$h_{33} = 0$

$h_{23} = 0.5\, h_{33} + 0.1\, h_{23} + 0.4\, h_{13} + 1$

$h_{13} = 0.5\, h_{33} + 0.5\, h_{23} + 1$

$$h_{13} = 0.5 \cdot h_{23} + 1$$

$$h_{23} = (0.5) \cdot 0 + 0.1 h_{23} + 0.4 (0.5 h_{23} + 1) + 1$$

$$= 0 + 0.3 h_{23} + 0.4 + 1$$

$$0.7 h_{23} = 1.4$$

$$\boxed{\begin{aligned} h_{23} &= 2 \\ h_{13} &= 2 \end{aligned}}$$

$$\pi \cdot P = \pi$$

$$(\pi_1, \pi_2, \pi_3) \cdot \begin{pmatrix} 0 & 1/2 & 1/2 \\ 2/5 & 1/10 & 1/2 \\ 3/5 & 2/5 & 0 \end{pmatrix} = (\pi_1, \pi_2, \pi_3)$$

1 $\pi_1 \cdot 0 + \pi_2 \cdot 2/5 + \pi_3 \cdot 3/5 = \pi_1$

2 $1/2 \pi_1 + 1/10 \pi_2 + 2/5 \pi_3 = \pi_2$

3 $1/2 \pi_1 + 1/2 \pi_2 = \pi_3$

4 $\pi_1 + \pi_2 + \pi_3 = 1$

1 -> 3 $\quad 1/2 \left(2/5 \pi_2 + 3/5 \pi_3 \right) + 1/2 \pi_2 = \pi_3$

$$1/5 \pi_2 + 1/2 \pi_2 + \frac{3}{10} \pi_3 = \pi_3$$

$$\frac{1}{5}\pi_2 + \frac{1}{2}\pi_2 + \frac{3}{10}\pi_3 = \pi_3$$

$$\frac{2+5}{10}\pi_2 = \frac{7}{10}\pi_3$$

$$\pi_2 = \pi_3$$

2: $$\frac{1}{2}\pi_1 + \frac{1}{10}\pi_2 + \frac{2}{5}\pi_2 = \pi_2$$

$$\frac{1}{2}\pi_1 + \frac{1+4}{10}\pi_2 = \pi_2$$

$$\frac{1}{2}\pi_1 = \frac{1}{2}\pi_2$$

$$\pi_1 = \pi_2$$

$$\pi_1 = \pi_2 = \pi_3 = \frac{1}{3}$$