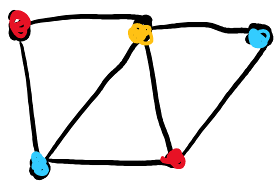
Tutorial X

09 May 2019 12:01

The probabilistic method II.

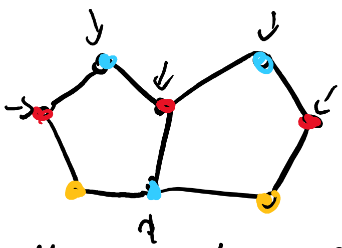
The existence of graphs with large girth (ℓ) and chromatic number.

Graph G has girth ℓ, if there are no cycles smaller than ℓ



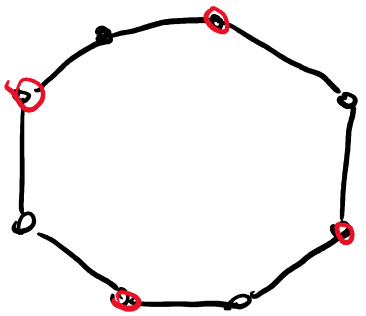
Girth = 3

$\chi(G) = 3$



Girth = 5

Chromatic number - the smallest number of colors for vertices such that no edge connects two vertices of the same color



$\alpha(G) = 4$

Independence number of graph G is the size of the largest independent set, which is a set of vertices without any edges

$$\alpha(G) \geq \frac{|V|}{\chi(G)} \qquad \chi(G) \geq \frac{|V|}{\alpha(G)}$$

Intuition why finding graphs with large chromatic number and large girth is difficult:

→ In order to avoid small cycles the number of edges is rather small

→ small number of edges leads to a large independence number

→ large independedence number leads to small chromatic number

We split the problem into two events

1.) the probability that the number of small cycles is large is smaller than $\frac{1}{2}$. E_1 - number of small cycles is large

2.) the probability of a large independence number is smaller than $\frac{1}{2}$. E_2 - independence number is large

We want a graph that has neither of these properties

$$\Pr(\neg E_1 \wedge \neg E_2) = 1 - \Pr(E_1 \vee E_2) \underset{\substack{\uparrow \\ \text{union bound}}}{\geq} 1 - (\Pr(E_1) + \Pr(E_2)) > 0$$

Random graph experiment - add each edge with probability p.

$p = n^{\lambda - 1}$ $\lambda \in (0, \frac{1}{\ell})$ [importantly $\lambda \cdot \ell < 1$]

We want probability that the number of small cycles is larger

We want probability that the number o s small cycles is larger than $\frac{n}{2}$ to be smaller than $\frac{1}{2}$.

Let X be the number of cycles smaller than ℓ.

$$\Pr\left(X > \frac{n}{2}\right) = ?$$

We are going to use Markov's inequality to get around the dependence of the cycles.

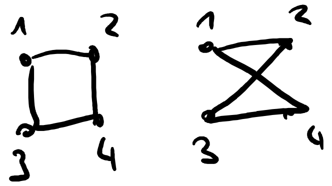
If $E(X) \leq \frac{n}{4}$, then by $\underbrace{\text{Markov's inequality}}_{\Pr(X>t) < \frac{E(X)}{t}}$ $\Pr\left(X > \frac{n}{2}\right) < \frac{1}{2}$

In order to calculate $E(X)$ define $N_{x_1,\dots,x_j}$ $= 1$ when vertices $x_1, \dots x_j$ form a cycle, $= 0$ otherwise

$$X = \sum_{j=3}^{\ell} \sum_{j\text{-tuples}} N_{x_1,\dots,x_j}$$

$$E(X) = \sum_{j=3}^{\ell} \sum_{j\text{-tuples}} \Pr(N_{x_1,\dots,x_j} = 1)$$

$$\Pr(N_{x_1,\dots,x_j} = 1) = p^j$$



$$\frac{\binom{n}{j} \cdot (j-1)!}{2}$$

$$E(X) = \sum_{j=3}^{\ell} \binom{n}{j} \frac{(j-1)!}{2} \cdot p^j$$

$$< \sum_{j=3}^{\ell} n^j p^j$$

$$= \sum_{j=3}^{\ell} n^j (n^{\lambda-1})^j$$

$$\boxed{\frac{n!\,(j-1)!}{j!\,(n-j)!\cdot 2} = \frac{n!}{2j\,(n-j)!} = \frac{n\cdot(n-1)\cdots(n-j+1)}{2j} < n^j \text{ for sufficiently large}}$$

$= \sum_{j=3}^{\ell} n^j (n^{\lambda-1})^j$

for sufficiently large n

$= \sum_{j=3}^{\ell} n^j n^{\lambda j} n^{-j}$ $\rightarrow p = n^{\lambda - 1}$

$= \sum_{j=3}^{\ell} n^{\lambda j}$

$< \sum_{j=0}^{\ell} n^{\lambda j}$ ← geometric series with gradient n^λ

$= \frac{1-(n^\lambda)^{\ell+1}}{1-(n^\lambda)}$

$= \frac{(n^\lambda)^{\ell+1}-1}{(n^\lambda)-1}$

$\leq \frac{n^{\lambda \ell} \cdot n^\lambda}{n^\lambda - 1}$

$= \frac{n^{\lambda \ell}}{1-n^{-\lambda}}$

$\frac{n^{\lambda\ell}}{1-n^{-\lambda}} < \frac{n}{c}$ (for each positive c)

$n^{\lambda \ell} < \frac{n}{c} \cdot (1-n^{-\lambda})$

$< \frac{n}{c} - \frac{n^{1-\lambda}}{c}$

$n^{\lambda\ell} + \frac{n^{1-\lambda}}{c} < \frac{n}{c}$

$\lambda \ell < 1$ $\quad 1-\lambda < 1$

n increases the fastest

$< \frac{n}{4}$

for sufficiently large n

$$\boxed{\begin{aligned} &E(X) < \tfrac{n}{4} \text{ for sufficiently large } n \\ &\Rightarrow Pr\left(X > \tfrac{n}{2}\right) < \tfrac{1}{2} \end{aligned}}$$

2.) Independence number $\alpha(G)$ is small

GOAL

$$\Pr(\alpha(G) \geq m) < \frac{1}{2}$$

$$\Pr(\alpha(G) \geq m) \overset{\text{Union bound}}{\leq} \sum_{S \subset V,\, |S| = m} \Pr(S \text{ is an independent set})$$

$$= \binom{n}{m} (1-p)^{\binom{m}{2}}$$

$$\binom{n}{m} \leq n^m$$

$$(1-p) < e^{-p}$$

$$< n^m e^{-p \frac{m(m-1)}{2}}$$

$$m = \left\lceil \frac{3}{p} \ln(n) \right\rceil$$

$$< n^m n^{-\frac{3}{p} \frac{(m-1)}{2}}$$

$$\left[e^{\ln n} = n \right]$$

$$= n^m n^{-3(m-1)/2}$$

$$= n^{m - \frac{(3m-3)}{2}} = n^{\frac{2m-3m+3}{2}}$$

$$= n^{\frac{3-m}{2}} \leq n^{\frac{3}{2} - \frac{3}{p} \ln(n)}$$

$$= n^{3/2 - \frac{3}{2} \cdot \frac{\ln(n)}{n^{(\lambda-1)}}}$$

$$\approx n^{-\frac{\ln(n)}{n^{-\epsilon}}} = \frac{1}{n^{\frac{\ln(n)}{n^{-\epsilon}}}} \overset{n \to \infty}{\leadsto} 0$$

Probability that independence number is larger than

$\lceil 3/p \ln(n) \rceil$ is smaller than $1/2$ for sufficiently large n.

Together the probability there is a graph G with the number of small cycles (smaller than ℓ) smaller than $\frac{n}{2}$ and independence number smaller than $\lceil \frac{3}{p} \ln(n) \rceil$ is positive. $\Rightarrow$ existence of such G

From G you can construct G' by deleting a vertex from each cycle.

$$\chi(G') > \frac{|V(G')|}{\alpha(G')} > \frac{n/2}{3 \cdot n^{1-\lambda} \cdot \ln(n)} \xrightarrow{n \to \infty} \infty$$

There are graphs with arbitrary girth ℓ and chromatic number χ.