Tutorial III

05 March 2020 14:02

TAIL INEQUALITIES

MARKOV'S INEQUALITY

CHEBYSHEV'S INEQUALITY

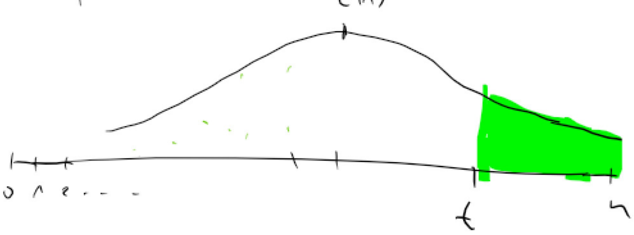
CHERNOFF'S INEQUALITY

MARKOV'S INEQUALITY

X - a random variable with positive values

$$\Pr(X \geq t) \leq \frac{E(X)}{t}$$

$t > E(X)$



EXAMPLE LV1 - always gives correct answer and expected number of calculationsteps $E(n)$ is polynomial in n (size of the input)

LV2 - alway has short calculations (polynomial in n) but gives correct answer w.p. p only w.p. $(1-p)$ it says IDK (??)

Run LV1 for some time $(2E(n)+1)$ and if calculation is too long stop and say '??'

LV2 algorithm,

The probability that our new algorithm produces '??'

X - number of steps needed in calculation for given input

$$\Pr(X \geq 2E(X)) \leq \frac{E(X)}{2E(X)} = \frac{1}{2}$$

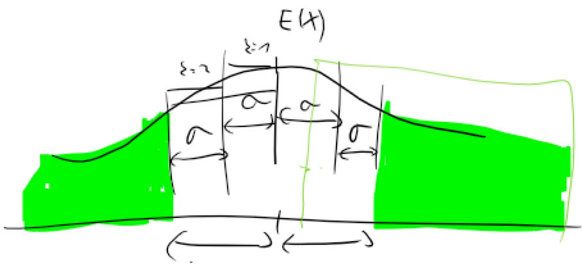
Chebyshev's inequality

X has a finite $E(X) = \mu$

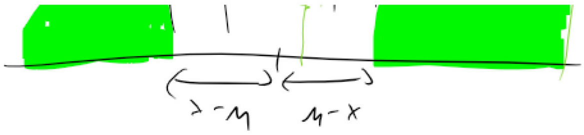
$Var(X) = \sigma^2$ σ - standard deviation

$$\Pr(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

$$\Pr(|X-\mu| \geq t) \leq \frac{\sigma^2}{t^2}$$



$$\Pr\left(|X-\mu| \geq t\right) \leq \frac{\sigma^2}{t^2}$$



$\boxed{\text{Useful only for } k>1}$

Chernoff's inequality

Specific form of random variables

$$X = \sum_{i=1}^{n} X_i, \quad \text{where } X_i \text{ are } \underline{\text{identically and independently distributed}}^{\text{i.i.d.}}$$

binary r.v.'s with $P(X_i=1) = p$.

$$\mu = E(X) = n \cdot p$$

Euler's number

$$\Pr\left(X > (1+\delta)\mu\right) < \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu}$$

$$\Pr\left(X < (1-\delta)\mu\right) < \left(\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}}\right)^{\mu}$$

HARD TO USE

Simpler (and looser) expressions are:

$$\Pr\left(X \leq (1-\delta)\mu\right) \leq e^{-\frac{\delta^2\mu}{2}} \leq e^{-\frac{\delta^2\mu}{3}} \qquad 0 \leq \delta \leq 1$$

$$\Pr\left(X \geq (1+\delta)\mu\right) \leq e^{-\frac{\delta^2\mu}{3}} \qquad 0 \leq \delta \leq 1$$

$$\Pr\left(|X-\mu| \geq \delta\mu\right) \leq 2e^{-\frac{\mu\delta^2}{3}}$$

$$\| $$

$$\Pr\left(X-\mu \geq \delta\mu\right) \vee \Pr\left(\mu - X \geq \delta\mu\right)$$

$$\left(X \geq \delta\mu + \mu\right)$$

$\boxed{\text{Exercise}}$

In the experiment we roll a 6-sided die n times.

Let X be the number of outcomes '6' in our experiment

Calculate (or estimate) $\boxed{\Pr\left(X \geq n/4\right) = \sum_{i \geq n/4}^{n} \binom{n}{i} \left(\frac{1}{6}\right)^{i} \left(\frac{5}{6}\right)^{n-i}}$

Markov's inequality

$E(X) = n/6 \qquad X = \sum_{i=1}^{n} X_i \qquad X_i = 1$ if ith roll results in '6'

$$E(X) \overset{(+)}{=} \sum_i E(X_i) = n \cdot E(X_i) = n \cdot \left(\tfrac{1}{6} \cdot 1 + \tfrac{5}{6} \cdot 0\right) = n/6$$

$$\Pr\left(X \geq \frac{n}{4}\right) \leq \frac{E(X)}{n/4} = \frac{n/6}{n/4} = 2/3$$

Chebyshev's inequality

$$Var(X) = n \cdot p \cdot (1-p) = n \cdot \tfrac{1}{6} \cdot \tfrac{5}{6} = \frac{5n}{36}$$

$$\boxed{\Pr\left(X \geq \frac{n}{4}\right)}$$

$$\Pr\left(|X - E(X)| \geq t\right) \leq \frac{Var(X)}{t^2}$$

$$\|$$

$$\Pr\left(X - E(X) \geq t\right) \vee \Pr\left(E(X) - X \geq t\right) \leq \frac{5n/36}{\frac{n^2}{144}} = \frac{20}{n}$$

$$\geq$$

$$\Pr\left(X - E(X) \geq t\right)$$

$$\|$$

$$\Pr\left(X \geq E(X) + t\right)$$

$$E(X) + t = n/4$$

$$t = n/4 - n/6$$

$$t = \frac{n}{12}$$

Chernoff's inequality

$$Pr(X \geq (1+\delta)\mu) \leq e^{-\frac{\delta^2 \mu}{3}}$$

$$(1+\delta)\mu = \frac{n}{4}$$

$$(1+\delta)\frac{n}{6} = \frac{n}{4}$$

$$\boxed{\delta = 1/2}$$

$$Pr\left(X \geq \left(1+\tfrac{1}{2}\right)\mu\right) \leq e^{-\frac{n}{72}}$$

Amplification of success probability for 2MC algorithms

$\boxed{BPP}$ – Probability of a correct result $\geq 1/2 + \varepsilon$

$\boxed{PP}$ – Probability of a correct result $> 1/2$ $\qquad 1/2 + \varepsilon(n)$

Probability amplification = run the algorithm k times and use majority voting

k

Chernoff's bound

X_i ~ characterizes i-th run

$X_i = 1$ if the correct answer was given

$X_i = 0$ if incorrect answer is given

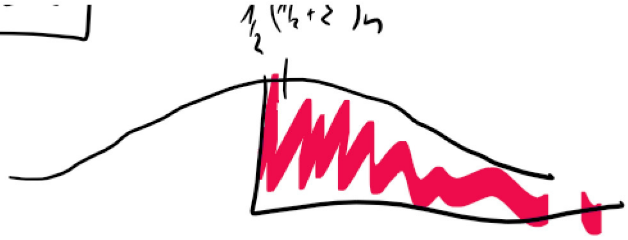
$X = \sum_{i=1}^{k} X_i$ is the number of correct answers

$E(X) = k \cdot (1/2 + \varepsilon)$ $\qquad \boxed{p = 1/2 + \varepsilon}$

$\frac{1}{2}(1/2 + \varepsilon)n$

$E(X) = k\cdot(1/2+\varepsilon)$

$\cancel{Pr(X \geq k/2)}$



$Pr(X \leq k/2)$

$Pr(X \leq (1-\delta)\mu) \leq e^{-\frac{\delta^2\mu}{2}}$

$(1-\delta)\mu = k/2$

$\vdots$

$\delta = \frac{\varepsilon}{1/2+\varepsilon} = \frac{\varepsilon}{p}$

$$Pr\Big(X \leq \big(1-\tfrac{\varepsilon}{p}\big)\cdot \overbrace{n\cdot p}^{\mu}\Big) \leq e^{-\frac{\overbrace{\frac{\varepsilon^2}{p^2}}^{\delta^2}\cdot \overbrace{k\cdot p}^{\mu}}{2}} = \dots = e^{-\frac{k\cdot\varepsilon^2}{1+2\varepsilon}}$$

$e^{-\frac{k\cdot\varepsilon^2}{1+2\varepsilon}} \leq \alpha$ ⇝ desired small error $\quad /\ln$

$-\frac{k\cdot\varepsilon^2}{1+2\varepsilon} \leq \ln\alpha$

$-k\varepsilon^2 \leq \ln\alpha\cdot(1+2\varepsilon)$

$k \geq -\frac{\ln\alpha\cdot(1+2\varepsilon)}{\varepsilon^2}$

let n be the size of the input.

→ if ε does not depend on n then ε does not depend on n either.

$\rightarrow$ let $\varepsilon = \frac{1}{\text{polynomial}(n)}$

then $\varepsilon(n)$ is still a polynomial

$\rightarrow$ let $\varepsilon = \frac{1}{2^n}$

$$k(n) \geq O\left(\frac{\frac{1}{2^n}}{\frac{1}{2^n \cdot 2^n}}\right) = O(2^n)$$