Tutorial IX

16 April 2020 13:54

MARKOV CHAINS I

- → basic definitions
- → hitting probabilities
- → hitting time
- → ERGODIC thm (stationary distributions)

Next week

- → walks on a line
- → 2 SAT
- → Fair 2-colorability of 3-colorable graphs

Definitions

Markov Chain (MC) is an infinite collection of r.v. $\{X_i\}_{i=0}^{\infty}$ with n outcomes, such that:

$$\forall i \quad \Pr(X_i = x_i \mid X_{i-1} = x_{i-1}, X_{i-2} = x_{i-2}, \dots, X_0 = x_0)$$

$$= \Pr(X_i = x_i \mid X_{i-1} = x_{i-1})$$

generally $\Pr(X_3 = 3 \mid X_2 = 2, X_1 = 1) \neq \Pr(X_3 = 3 \mid X_2 = 2, X_1 = 2)$

INTERPRETATION:

MC's are the simplest non-trivial stochastic processes. Each X_i is a state of the process in time step i, The process can have

n states. Markov property says that the state of the process in step $i+1$ depends only on state in time i and the whole past.

This leads to simplification:

$$\Pr(X_3=3 \mid X_2=2) = \Pr(X_7=3 \mid X_6=2) = \Pr(X_{i+1}=3 \mid X_i=2)$$

$$= P_{23}$$ (probability of moving from state 2 to state 3.)

How many probabilities describe MC? $n \times n$

Matrix representation

Transition matrix P

$$P = \begin{pmatrix} P_{11} & P_{12} & P_{13} & \cdots & P_{1n} \\ P_{21} & P_{22} & P_{23} & \cdots & P_{2n} \\ \vdots & & & \ddots & \vdots \\ P_{n1} & \cdots & & & P_{nn} \end{pmatrix}$$

This matrix is stochastic

$$\forall i \quad \sum_j P_{ij} = 1$$

If $X_0 = \ell$ (if I start the process in state ℓ)

$\Pr(X_k = 1) = ?$

ℓ-th position

1.) $(0,0,\dots,1,\dots,0,0)\cdot P = (p_{\ell 1}, p_{\ell 2}, \dots, p_{\ell n})$

2.) $(p_{\ell 1},\dots,p_{\ell n})\cdot P = (0,0,\dots,1,\dots,0,0)\,P^2$

in k-steps

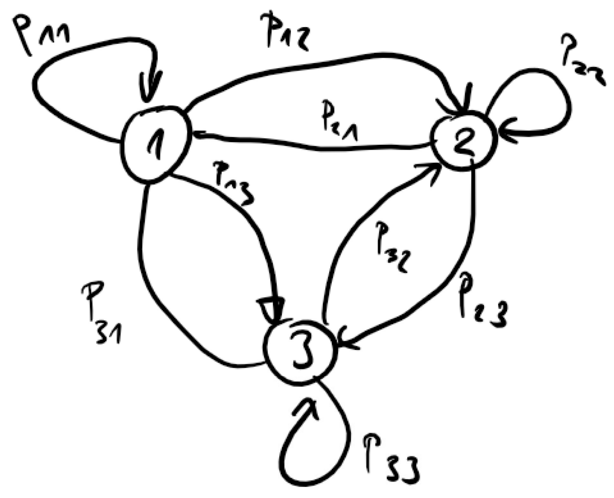
k.) $(0,0,\dots,1,\dots,0)\cdot P^k$

P^k - k-step transition matrix

GRAPH REPRESENTATION

Graph with n vertices (each vertex corresponds to a state) and directed edges labelled by transition probabilities

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$



Basic properties

$r_{ij}^{(t)}$ ~ the probability to reach j from i in exactly t-steps

Hitting probability - the over all probability to reach j from i

$$f_{ij} = \sum_{t=0}^{\infty} r_{ij}^{(t)}$$

Hitting time - average time to reach j from i

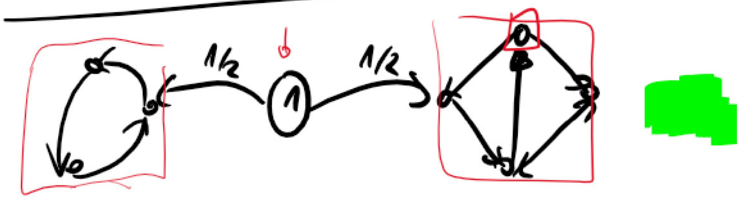
$$h_{ij} = \sum_{t=0}^{\infty} t \cdot r_{ij}^{(t)}$$

Ergodic theorem (Stationary distribution)

For an ERGODIC MC (all states can reach all other states w.p. 1 and MC is not periodic), there exists a unique probability vector $\pi = (\pi_1, \dots, \pi_n)$, such that

$$\pi = \pi P$$

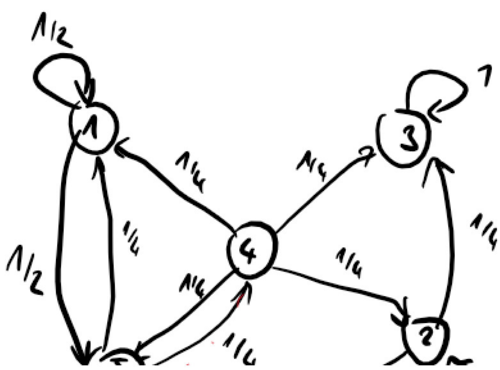
and for all j and p: $\pi_j = \lim_{k \to \infty} p P^k$



Periodic MC:

EXERCISES:

$$P = \begin{pmatrix} 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/4 & 0 & 1/4 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$



$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 0 & 1/4 \\ 1/4 & 0 & 0 & 1/4 & 1/2 \end{pmatrix}$$

TASK: Caculate f_{i4} for each i

$$\boxed{\begin{aligned} f_{44} &= 1 \\ f_{14} &= 1/2\, f_{14} + 1/2\, f_{54} \Rightarrow f_{14} = f_{54} \\ f_{24} &= 1/4\, f_{34} + 1/2\, f_{24} + 1/4\, f_{54} \\ f_{34} &= f_{34} \qquad \boxed{f_{34} = 0} \\ f_{54} &= 1/4 \cdot f_{14} + 1/4 \cdot f_{44} + 1/2\, f_{54} \end{aligned}}$$

5 equation with 5 variables

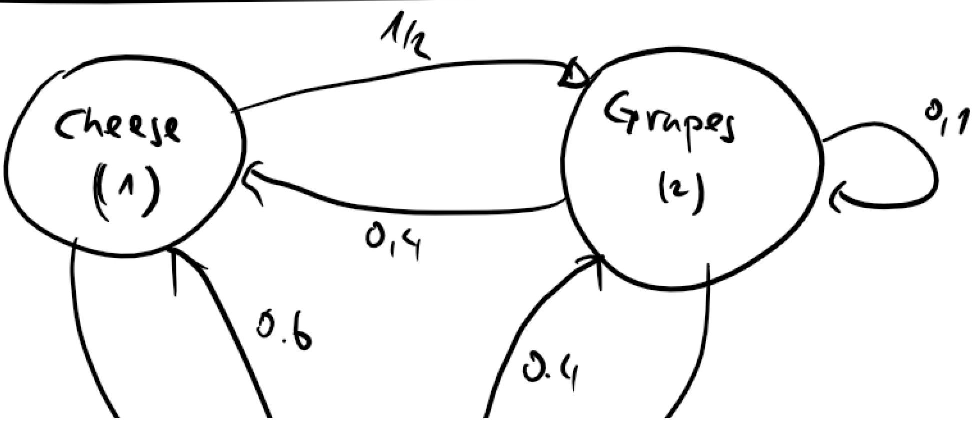
$$f_{54} = 1/4\, f_{54} + 1/4 + 1/2\, f_{54}$$

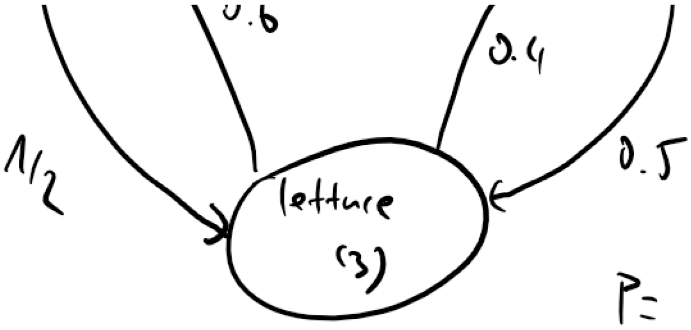
$$1/4\, f_{54} = 1/4 \Rightarrow f_{54} = 1 \text{ and } f_{14} = 1$$

$$f_{24} = 0 + 1/2\, f_{24} + 1/4$$

$$f_{24} = 1/2$$

$$\boxed{f_{i4} = (1, 1/2, 0, 1, 1)}$$





$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 0.4 & 0.1 & 0.5 \\ 0.6 & 0.4 & 0 \end{pmatrix}$$

Calculate f_{i3}

$f_{33} = 1$

$f_{23} = 0.5 f_{33} + 0.1 f_{23} + 0.4 f_{13}$

$f_{13} = 0.5 f_{33} + 0.5 f_{23} \Rightarrow f_{13} = \frac{1}{2} f_{23} + \frac{1}{2}$

$f_{23} = 0.5 + 0.1 (f_{23}) + 0.4 \left(\frac{1}{2} f_{23} + \frac{1}{2}\right)$

$\frac{1}{2} + \frac{1}{5} + 0.3 f_{23}$

$0.7 f_{23} = \frac{7}{10}$

$f_{23} = 1 \Rightarrow f_{13} = 1 \qquad f_{i3} = 1$

$\boxed{\forall_{ij} f_{ij} = 1} \Rightarrow$ Ergodic

Calculate h_{i3}

$$h_{33} = 0$$

$$h_{23} = \tfrac{1}{2} h_{33} + \tfrac{1}{10} h_{23} + \tfrac{4}{10} h_{13} + 1$$

$$h_{13} = \tfrac{1}{2} h_{33} + \tfrac{1}{2} h_{23} + 1 \Rightarrow h_{13} = \tfrac{1}{2} h_{23} + 1$$

$$h_{23} = \tfrac{1}{2}\cdot 0 + \tfrac{1}{10} h_{23} + \tfrac{4}{10}\left(\tfrac{1}{2} h_{23} + 1\right) + 1$$

$$= 0 + 0.3\, h_{23} + 0.4 + 1$$

$$0.7\, h_{23} = 1.4$$

$$h_{23} = 2 \qquad\qquad h_{13} = \tfrac{1}{2}\cdot 2 \cdot 1 = 2$$

$$\pi \cdot P = \pi$$

$$(\pi_1, \pi_2, \pi_3)\begin{pmatrix} 0 & 1/2 & 1/2 \\ 0.4 & 0.1 & 0.5 \\ 0.6 & 0.4 & 0 \end{pmatrix} = (\pi_1, \pi_2, \pi_3)$$

1. $\pi_1 \cdot 0 + \pi_2 \cdot 2/5 + \pi_3 \cdot 3/5 = \pi_1$
2. $\pi_1 \cdot 1/2 + \pi_2 \cdot \frac{1}{10} + \pi_3 \cdot 2/5 = \pi_2$
3. $\pi_1 \cdot 1/2 + \pi_2 \cdot 1/2 + \pi_3 \cdot 0 = \pi_3$
4. $\pi_1 + \pi_2 + \pi_3 = 1$

$1 \to 3$

$$\tfrac{1}{2}\left(\tfrac{2}{5}\pi_2 + \tfrac{3}{5}\pi_3\right) + \tfrac{1}{2}\pi_2 = \pi_3$$

$$\tfrac{1}{5}\pi_2 + \frac{3}{10}\pi_3 + \tfrac{1}{2}\pi_2 = \pi_3$$

$$\frac{2+5}{10}\pi_2 = \frac{7}{10}\pi_3$$

$$\pi_2 = \pi_3 \to ?$$

$$\tfrac{1}{2}\pi_1 + \tfrac{1}{10}\pi_2 + \tfrac{2}{5}\pi_2 = \pi_2$$

$$\frac{1}{2}\pi_1 = \frac{5}{10}\pi_2$$

$$\pi_2 = \pi_1$$

4: $\Rightarrow \pi_1 = \pi_2 = \pi_3 = 1/3$

$$(1/3, 1/3, 1/3) \cdot P = (1/3, 1/3, 1/3)$$