Tutorial V

19 March 2020 14:03

→ Game theory

→ Lower bounds on efficiency of Randomized Algorithms

→ Example: Tree evaluation

$$\begin{array}{cc|ccc} & & & \text{Bob} & \\ & & R & P & S \\ \hline & R & 0 & \boxed{-1} & 1 \\ \text{Alice} & P & 1 & 0 & \boxed{-1} \\ & S & \boxed{-1} & 1 & 0 \end{array}$$

→ Game evaluation matrix

→ Alice is trying to maximize the outcome

→ Bob is trying to minimize the outcome

Generally this GEM is $[M_{ij}]$ of real numbers

If Alice chooses strategy i, in the worst case she gets

$$\min_j [M_{ij}]$$

What is Alice's best strategy? $\max_i \min_j M_{ij} = O_A$

What is Bob's best strategy? $\min_j \underbrace{\max_i M_{ij}}_{\substack{\text{guaranteed}\\\text{outcome}\\\text{of strategy } j}} = O_B$

There are games for which $O_A = O_B$

equilibrium

$$\begin{array}{c|ccc} & & & \text{B} \\ \hline & 0 & -1 & \boxed{-2} \\ \text{A} & 1 & 0 & \boxed{-1} \\ & \boxed{2} & \boxed{1} & \boxed{0} \end{array}$$

MIXED STRATEGIES

Alice's strategy = probability distribution over rows p

Bob's strategy = probability distribution over columns q

p, q – column vectors of probabilities

$$p^T M q = \sum_{ij} p_i q_j M_{ij} = \text{Expected value of game } M \text{ with strategies } p \text{ and } q$$

For fixed strategy of Alice p she is guaranteed to achieve value of at least $\min p^T M q$

For fixed strategy of Alice p she is guaranteed to achieve value of at least $\min_q p^T M q$

Alice's best strategy $\max_p \min_q p^T M q = O_A$

Bob's best strategy $\min_q \max_p p^T M q = O_B$

Von Neumann's theorem (Von Neumann's equilibrium)

$$\forall M \quad \max_p \min_q p^T M q = \min_q \max_p p^T M q$$

Loomis' theorem

$$\max_p \min_k p^T M e_k = \min_q \max_j e_j^T M q$$

$e_i = (0, \dots, 1, \dots, 0)$ — i^{th} position

$a^T = p^T M$

$\boxed{\text{Proof sketch}}$

For fixed p: $p^T M q = a^T q = a_1 q_1 + \dots + a_n q_n$

find the smallest a_i
and choose $q_i = 1$ and others $= 0$

$a^T q$ is linear in q and $\sum_i q_i = 1$.

Recall that randomized algorithms can be seen as mixtures of deterministic algorithms.

deterministic algorithms which randomized algorithm chooses from

Bob

$$\begin{array}{c|ccccc}
 & A_1 & A_2 & A_3 & \dots & A_n \\
\hline
I_1 & C(I_1,A_1) & C(I_1,A_2) & \dots & & \\
I_2 & & & & & \\
I_3 & & & & & \\
\vdots & & & & & \\
I_m & & & & &
\end{array} \; = M$$

Alice

$E(C(I_p, A_q))$ = expected running time for input distribution p and rand. algorithm characterized by probability q.

$= p^T M q$.

by probability q.

$$= p^T M q.$$

vN's thm:

$$\max_p \min_q E(C(I_p, A_q)) = \min_q \max_p E(C(I_p, A_q))$$

Loomis thm:

$$\cancel{\max_p} \min_{A_i \in \mathcal{A}} E(C(I_p, A_i)) = \cancel{\min_q} \max_{i \in I} E(C[I_i, A_q])$$

$$\forall_{p,q} \quad \underbrace{\min_{A_i \in \mathcal{A}} E(C(I_p, A_i))}_{} \leq \underbrace{\max_{i \in I} E(C[I_i, A_q])}_{}$$

for a chosen input distribution find the best deterministic algorithm

a lower bound!

for given algorithm (randomized) find the worst input

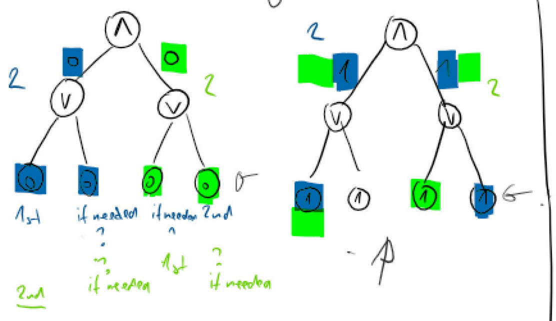
we are interested in this!

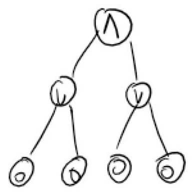
Tree evaluation example

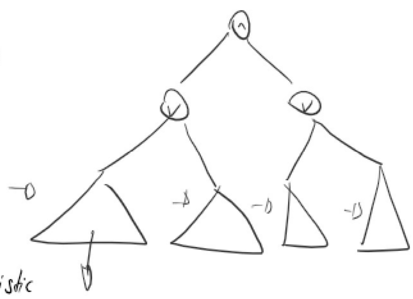
Example input distribution

all 0's w.p. 1/2 ~ (all nodes evaluate to 0)
all 1's w.p. 1/2 ~ (all nodes evaluate to 1)

Deterministic alg







uniform over these 8 deterministic ones

first node to evaluate - 4 choices
choice of the node - 2 choices
In total 8 deterministic algorithms for every level

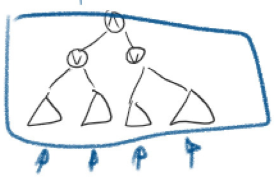
this subtree evaluates to
0 w.p. 1/2
and 1 w.p. 1/2

$\Downarrow$

Any randomized algorithm needs at least 2 evaluations for each level $\Rightarrow$ 2^k complexity with k levels,

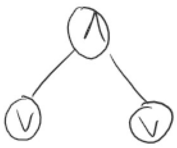
Example distribution 2:

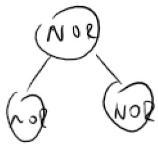
Example distribution?:



1 1 0 0

0 0 1 1





$(a \vee b) \wedge (c \vee d)$

$\|$

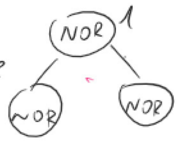
$(a \text{ NOR } b) \text{ NOR } (c \text{ NOR } d)$

ab	a NOR b
00	1
01	0
10	0
11	0

$Pr(\text{leaf} = 1) = p$

$Pr(\text{leaf} = 0) = (1-p)$

$p = (1-p)^2$ then the probability of a tree

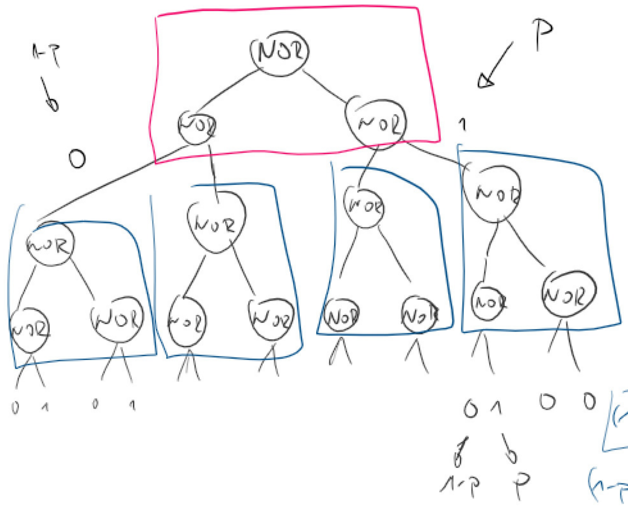


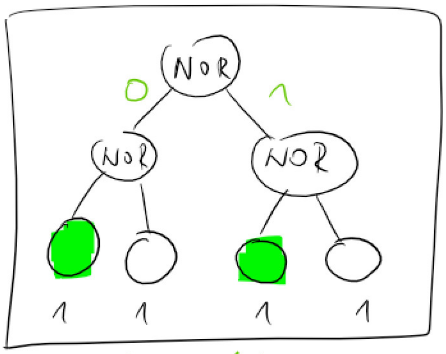
evaluating to 1

$p = \frac{3-\sqrt{5}}{2}$

is P and to 0 $1-p$.

H is recursive!





	2nd	?	1st	?	2	Prob
$(1-p)^3 p$	0	0	0	1	4	0,05
$(1-p)^3 p$	0	0	1	0	3	0,05
$(1-p)^2 p^2$	0	0	1	1	3	0,05
	0	1	0	0	2	0,05

$$\sum_{input} p(input) \cdot h(input) \geq 2$$

$p = \frac{3-\sqrt{5}}{2} \approx 0{,}38$ $\quad (1-p)^3 = 0{,}24$

$p^2 \approx 0{,}15$ $\quad \boxed{0{,}038} = (1-p)^7$

$p^3 \approx 0{,}06$ $\quad (1-p) = 0{,}62$

$p^4 \approx 0{,}02$

$(1-p)^4 = p^2 < 0{,}14$

$p \cdot (1-p)^3 < 0{,}09$

$p^2 \cdot (1-p^2) = p^3 < 0{,}05$

$p^3 (1-p) < 0{,}03$

$p^4 < 0{,}02$

always rounding down.

0	1	0	1	4	0,05
0	1	1	0	3	0,05
0	1	1	1	3	0,03
1	0	0	0	2	0,09
1	0	0	1	3	0,05
1	0	1	0	2	0,05
1	0	1	1	2	0,03
1	1	0	0	2	0,05
1	1	0	1	3	0,05
1	1	1	0	2	0,03
1	1	1	1	2	0,02
0	0	0	0	2	0,14

Expected number of steps: $4 \cdot (0{,}09 + 0{,}05) + 3 \cdot (0{,}09 + 4 \cdot 0{,}05 + 0{,}03)$

$+ 2 \cdot (2 \cdot 0{,}09 + 2 \cdot 0{,}05 + 2 \cdot 0{,}03 + 0{,}14 + 0{,}02)$

$\|$

$0{,}56 + 0{,}96 + 1$

$\|$

$2 < 2{,}52 < 3$

lower bound with all 0 or all 1 inputs

upper bound from tutorial IV

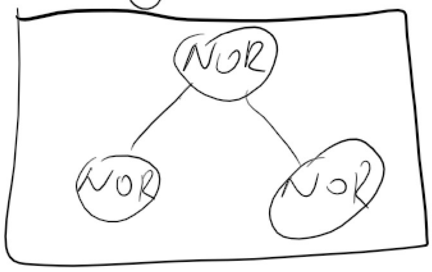
Easier derivation consistent with slides:

$Pr(i_1=1)=p$ $\quad$ $Pr(i_2=1)=p$

Need to examine only one input

need to examine both inputs

each NOR will take exactly $p + (1-p)\cdot 2 = (2-p) < 1{,}61$ steps to evaluate. Therefore



takes roughly

$$p\cdot(2-p) + (1-p)\cdot 2\cdot(2-p) = (2-p)^2 < 2{,}61 \text{ steps to evaluate}$$

Finally we have

$$2 < 2{,}52 < 2{,}61 < 3$$

this is a bit lower due to probability rounding down