Tutorial VIII

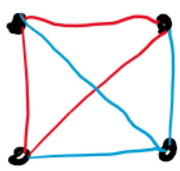
09 April 2020 14:06

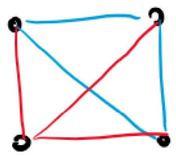
PROBABILISTIC METHOD

Ramsey number

Ramsey number $R(k,t)$ is the smallest number n, such that each 2-coloring of edges of K_n (complete graph of n vertices) has a red subgraph K_k or blue subgraph K_t.

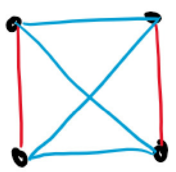
R(3,3)



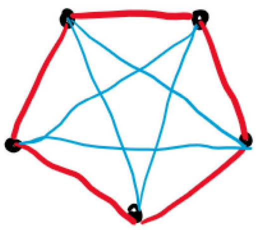


How many colorings does K_4 have?

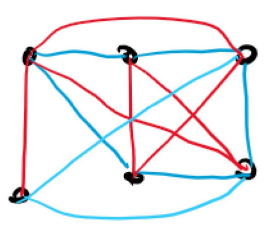
$2^6 = 64$



$\rightarrow$ no red or blue triangle $\Rightarrow R(3,3) > 4$



$\Rightarrow R(3,3) > 5$



How many colorings are there?

$\binom{6}{2}$ - edges $\frac{6!}{2!\,4!} = \frac{5 \cdot 6}{2} = 15$ edges

$2^{15} \approx 32$ million colorings

$R(3,3) = 6$

PROBABILISTIC ARGUMENT

→ Color the graph "randomly" and if probability of a counterexample is larger than 0, the counter example <u>must</u> exist ⇒ Lower bounds on Ramsey number.

Thm. from slides

$$\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1 \quad \Rightarrow \quad R(k,k) > n$$

Let us consider the following coloring experiment:

Color every edge of K_n red w.p. $1/2$
blue w.p. $1/2$

Choose $S \subset V$, $|S| = k$

$r_s = 1$ if graph induced by S is all red

$b_s = 1$ if graph induced by S is all blue

$$\forall_S \; Pr(r_s = 1) = \frac{1}{2}^{\binom{k}{2}}$$

$$\forall_S \; Pr(b_s = 1) = \frac{1}{2}^{\binom{k}{2}}$$

$$\forall s \quad Pr(b_s = 1) = \frac{1}{2}^{\binom{k}{2}}$$

$$Pr(r_s \vee b_s) = 2 \cdot \frac{1}{2}^{\binom{k}{2}} = 2^{1-\binom{k}{2}}$$

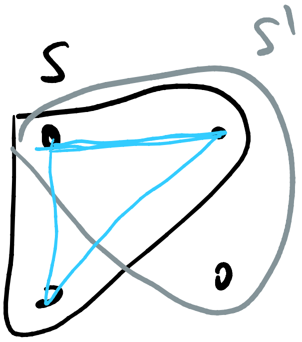
↳ S is either red or blue (monocolored)

What is the probability of some $S \subset V$ $|S| = k$ to be monocolored?

$$Pr\left(\bigvee_{S \subset V,\, |S| = k} r_s \vee b_s\right) < \binom{n}{k} \cdot 2^{1-\binom{k}{2}}$$

$\leq$

because r_s and $r_{s'}$ are generally not independent



$$1 - P\left(\bigvee_{S \subset V,\, |S| = k} r_s \vee b_s\right)$$

is a probability that graph contains no monocolored subgraph of size k

= COUNTEREXAMPLE

We need

$$1 - P(\bigvee_{\dots} v_s \vee b_s) > 0$$

$$\Downarrow$$

$$P(\bigvee v_s \vee b_s) < 1$$

$$P(\bigvee_{\dots} v_s \vee b_s) < \binom{n}{k} 2^{1-\binom{k}{2}} < 1$$

$$\boxed{\text{if } \quad n = \lfloor 2^{\frac{k}{2}} \rfloor \text{ then } \quad R(k,k) \geq n}$$

Plug $\lfloor n = 2^{\frac{k}{2}} \rfloor$ into our thm and see if $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$

over estimate

$$\binom{n}{k} 2^{1-\binom{k}{2}} \leq \frac{n^k}{k!} 2^{1-\frac{k(k-1)}{2}}$$

$$< \frac{\left(2^{\frac{k}{2}}\right)^k}{k!} \cdot \frac{2}{2^{\frac{k(k-1)}{2}}}$$

$$= \frac{\cancel{2^{\frac{k^2}{2}}}}{k!} \cdot \frac{2}{\cancel{2^{\frac{k^2}{2}}} \cdot 2^{-\frac{k}{2}}}$$

$$= \frac{2 \cdot 2^{\frac{k}{2}}}{k!} = \frac{2^{1+\frac{k}{2}}}{k!}$$

$k = 3$

$$\frac{2^{1+\frac{3}{2}}}{6} = \frac{2^{3/2}}{3} = \frac{\sqrt{8}}{3} \approx 0.9\dots$$

$$\frac{\;}{6} = \frac{\;}{5} = \frac{\;}{3} \approx 0.9\ldots$$

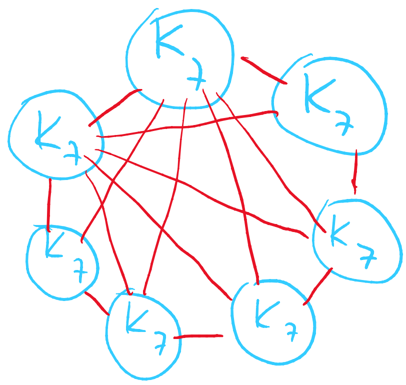
$k=4$ $$\frac{2^{1+2}}{4!} = \frac{8}{24} = \frac{1}{3}$$

$R(3,3) > \lfloor 2^{\frac{3}{2}} \rfloor = \lfloor \sqrt{8} \rfloor = 2$ (real $R(3,3)=6$)

$R(4,4) > \lfloor 2^{\frac{4}{2}} \rfloor = 4$

$R(8,8) > \lfloor 2^{4} \rfloor = 16$

$R(8,8) > 49$



$\Rightarrow$ no monocolored K_8

$R(k,k) > (k-1)^2$ constructive (k blue K_{k-1} graphs connected by red edges)

$> 2^{\lfloor \frac{k}{2} \rfloor}$ probabilistic

$\boxed{\text{Slides}}$ $R(3,3)=6$

$R(4,4)=18$

$43 \leq R(5,5) \leq 49$

Thm

if $\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{t} (1-p)^{\binom{t}{2}} < 1$

then $R(k,t) \geq n$ $\quad (0 \leq p \leq 1)$

RE: color each edge blue w.p. p
red w.p. $1-p$

$B \subset V, |B| = k$

$Pr(B \text{ is all blue}) = p^{\binom{k}{2}}$

$Pr(\text{some subset of size } k \text{ is all blue}) < \binom{n}{k} p^{\binom{k}{2}}$

$R \subset V, |R| = k$

$Pr(R \text{ is all red}) = (1-p)^{\binom{t}{2}}$

$Pr(\text{some subset of size } t \text{ is all red}) < \binom{n}{t} (1-p)^{\binom{t}{2}}$

Probability of a counterexample for $R(k,t)$

$$\boxed{1 - \binom{n}{k} p^{\binom{k}{2}} - \binom{n}{t} (1-p)^{\binom{t}{2}}}$$

we need this to be larger than 0.

$R(4,t)$

$\binom{n}{4} p^6 + \dots$

$\frac{n^4}{4!} p^6$

$\wr$

$1/24$

$$\left(\boxed{p = n^{-2/3}} \right)$$